Dynamical Systems, Individual-Based Modeling, and Self-Organization

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Contents:

Summary

To understand the workings of systems from diverse areas of research, e.g. ecology, economy, the social sciences, and biology, it is important to take into account the interactions between the individual elements of the system under study. This is especially true when the behavior of the system is a form of *self-organization*, the development of order that is not caused by external organizing forces, but a result of interactions between the elements of the system. This realization has lead to the use of individual-based models. Computer simulations of such models allow the investigator to test different hypotheses about the system under study.
The mathematical theory for such systems and systems that change over time in general, is that of dynamical systems. Valuable concepts have been developed to increase understanding of such systems. This article explains the basis concepts of dynamical systems theory, and shows their role in examples from physics, chemistry, biology, ecology, traffic, and the brain.

Introduction

Some systems in this world are best understood by considering the parts they consist of and the possible interactions between them, and combining these to infer the expected behavior of the system. Examples of such systems are the cantilever, the pulley, and the mechanical clock.

As soon as the number of elements in a system becomes more than a few however, this approach becomes infeasible. The classical approach then is to measure global properties of the system. A good example of this is thermal energy. Whereas it would be virtually impossible to measure the energy of each individual particle of a substance, it is straightforward to measure the temperature of the substance. If the substance is in a state of thermodynamic equilibrium, its thermal energy can be computed from the temperature.

The principle of using a global measure (temperature, in the example) as a characteristic for the behavior of the complete system is used in almost all sciences. Some other examples are indices of stock exchanges in economy, mortality and growth rates in ecology, and welfare statistics in sociology.

The validity of such global measures stands or falls with the presence of equilibrium. Measures of systems that are not near equilibrium are unlikely to be representative of the complete system. Moreover, the development over time of non-equilibric systems can differ widely from that of the same system near equilibrium, as will be seen in this article.

Growing awareness of these and related issues, in combination with the widespread availability of computers, has lead researchers in many different fields to use different types of models. Next to the equilibrium-based models that have been common for long, other models begin to emerge that explicitly represent the individual elements in the system of interest. Such models allow researchers to study the effects of the local interactions between these elements, and to understand how global behavior can be highly unpredictable as a result of these interactions.

The mathematical theory appropriate for systems whose behavior is determined by interacting elements is that of dynamical systems. This branch of mathematics studies systems that change over time, and has developed concepts that may improve understanding of the real world systems that are modeled. This article will introduce some of the basic concepts of this theory, and then give examples of real world systems that exhibit self-organization and chaos.

1. Individual-based modeling

One of the clearest examples where modeling the individual elements of a system has been recognized and used as an alternative to classic, equilibrium based models, is ecology. Here, measuring and modeling the individual elements of a system instead of global properties of the system is known as *individual-based modeling*.

To convey the point that modeling interactions between individuals is required for understanding global behavior of the system, an example concerning ethological research into the foraging behavior of ants will be briefly described. Ants transporting food tend to use the same routes. The choice of these routes is not given in by a central coordinating force, nor by any individual ant. Rather, this choice is the result of interactions between the ants through the environment. While walking, ants deposit a pheromone. After food has been found, the amount of pheromone that is deposited during walking increases. Since ants are attracted by this substance, they are more likely to walk over pheromone trails than in other places. While differences in the amounts of pheromone on different paths are initially small, these differences are enlarged by the fact that the paths that already have more pheromones attract more ants, and thus gather even more pheromones. This positive feedback principle increases initially small differences such that eventually all ants follow the same path. Due to the increase in deposits caused by finding food, and due to the shorter travel times and hence higher frequencies of shorter paths, self organization causes ants to find short paths to food sources.

Individual-based modeling has been used to investigate the relative influences of space perception, memory, pheromone trails, sensitivity of the sensory system, stimulation of nestmates, and food distribution on foraging. By measuring the values of parameters in the subject of study, such as the speed of ants with and without food, variance in heading distribution, and pheromone deposition interval, realistic simulations are produced that allow researchers to test hypotheses about the behavior of the animals.

The behavior of ants is only one example where individual interactions are important in understanding the global behavior of a system. Topics that have been investigated with individual models include economy, anthropology, sociology, linguistics, and ecology research on mammals, fish, birds, insects, bacteria, mixed ecosystems, and forests. Another example of an individual based model described later in this article concerns traffic flow.

Individual-based models are interesting in that they allow investigating the effect of interactions among the elements of a system. However, the dynamical systems perspective can also provide insight when used to model the dynamics of higher level properties of a system, such as population size, in cases where classic, equilibrium based models are not applicable. Apart from stable equilibria, observed fluctuations of population dynamics have been explained using periodic cycles, quasi-periodic cycles, and chaos. An example of this is given by the population dynamics of the flour beetle. These have been shown to be subject to a chaotic regime. Using a nonlinear demographic model, the dynamics under laboratory conditions were first predicted, to determine when chaotic should be expected. Testing this prediction experimentally confirmed the transition to chaos.

These results suggest that human intervention in ecological systems demands a firm understanding of the population dynamics, since the effects of actions can widely differ from expectations when chaotic population dynamics are mistakenly assumed to be approximately linear.

2. Basic notions of dynamical systems theory

In order to understand emergence and self-organization, it is necessary to study some basic notions of dynamical systems theory.
2.1 dynamical system *Dynamical systems* are systems that change over time. Only in systems that change over time can emergence and self-organization take place. All processes that can occur in nature can be described as dynamical systems, albeit usually very complex dynamical systems.
2.2 State variable The state of a dynamical system can be described by a number of *state variables*.
2.3 Dimension The minimum number of variables that completely captures the state of the system is referred to as the *dimension* of the system. An example of a simple dynamical system is that of a pendulum that swings in a plane. In the case of the pendulum, the state variables are the pendulum's position and its velocity. The pendulum is therefore a two-dimensional system.
Apart from the state variables, there can be other factors that can influence the behavior of dynamical systems. However, these do not change over time and are therefore not part of the system's state.
2.4 Control parameter These other factors are called the *control parameters* of the system. In the example of the pendulum, the control parameters are the length of the pendulum, the friction and the strength of gravity.

2.5 Control law Finally, a dynamical system is governed by *control laws*. Control laws determine the next state of the system for any given state. In the pendulum example, the control law is determined by Newton's second law of motion.

The state variables of a dynamical system by definition give a complete description of the state of the system.

2.6 Determinism Furthermore, in a *deterministic system*, the control laws always give the same successor state for any chosen state. Thus, there are no random influences, and the state of the system at any point in time completely determines the future behavior of the system. In real world systems however, it is generally impossible to determine the values of the state variables exactly, since this demands infinite precision of the measurements. In addition, the laws governing the system may be such that it is impossible to compute the future states of the system exactly, even if they are determined. This is the case in the *n-body problem* for instance, for $n \geq 3$.

2.7 Non-determinism Systems in which there are random influences also exist. Such systems are called *non-deterministic* or *stochastic*, and it is in general not possible to predict their future behavior exactly. For these systems, it is not possible to find a description with more state variables such that the system becomes deterministic. In other words, the non-determinism is not caused by hidden state variables. However, very complicated deterministic dynamical systems will often be modeled as non-deterministic systems with fewer state variables, in which case the non-determinism is caused by hidden state variables.

In order to make a mathematical description of a dynamical system it is necessary to decide whether to model time as continuous, or as divided into discrete slices. Physical systems will generally be modeled using continuous time, but for simulation purposes it is useful to create computer models that work with discrete time. In the case of continuous time, the equation of any dynamical system can be written as follows:

$$\dot{\mathbf{x}} = f(\mathbf{x}, t), \tag{1}$$

where $\mathbf{x}$ is the vector of state variables, f is the function that describes the behavior of the system and t is time.

2.8 Differential equation This is a so-called *differential equation*. The notation with bold face for vectors and overdots for derivatives is standard in dynamical systems literature.

2. 9 Difference equation In the case of discrete time, the equation is as follows:

$$\mathbf{x}_{t+1} = f(\mathbf{x}_t, t) \tag{2}$$

which is a *difference* equation.

2.10 Autonomous dynamical systems The above equations are of the most general form. Many dynamical systems that are encountered in practice have control laws that are independent of time, so that the value of f only depends on the state variables $\mathbf{x}_t$. Dynamical systems of this kind are called *autonomous* dynamical systems.

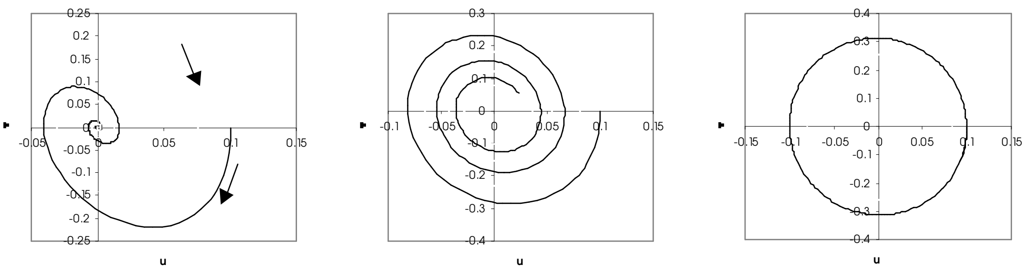

Figure 1: Phase plots of the pendulum.

Example 2.1: Pendulum

A pendulum can best be described in continuous time. Its state variables are its position *u* and its velocity *v*. The functions approximating its behavior for small angles are:

$$\dot{u} = v$$
$$\dot{v} = -g\frac{u}{l} - bv \qquad (3)$$

where *g* is the acceleration due to gravity, *l* is the length of the pendulum and *b* is a factor representing friction. It can be observed that these equations do not depend on time (the pendulum is therefore an autonomous dynamical system) and that the state variables occur only in linear combinations.

2.11 Linear dynamical systems Equation 3 is a *linear* dynamical system. The advantage of linear dynamical systems is that they can be solved analytically. This means that expressions for its variables can be found that depend only on time, and hence that the system's state for any point in the future can be calculated directly. This is not generally true for non-linear dynamical systems, and it will be shown below that only non-linear dynamical systems can show emergent and self-organizing behavior.

However, for now it is convenient to stick to the example of the pendulum in order to illustrate an important way of representing dynamical systems graphically.

2.12 Phase plot This is by means of a *phase plot* of the behavior of the system. A phase plot is a plot of the state variables of a system against each other.

2.13 State space (Phase space) The space of all possible system states is called the *state space* or *phase space* of the system, hence the name phase plot. The phase plot shows at a glance what paths connect the different states.

2.14 Trajectories Such paths are called *trajectories*, and depict what states the system will pass through when starting in a given state. Unfortunately, many of the more interesting dynamical systems have more than two degrees of freedom, and their phase spaces are impossible to plot exactly. In these cases, projections of the phase space may provide insight into the system's behavior.

Example 2.2: Phase plots of the pendulum

Examples of phase plots of the pendulum system are given in figure **Fout! Verwijzingsbron niet gevonden.**. In the leftmost frame, the phase plot of a strongly damped pendulum with a length of one meter is shown. Two trajectories are shown, beginning at different positions and velocities. The arrows indicate the direction in which the system evolves. Different trajectories can not cross in a deterministic system.

In the second frame, trajectories of a pendulum with equal length, but with less damping are shown. This shows the role a control parameter can play. Although the behavior of the system is different, it is not qualitatively different. All trajectories in the system still converge towards a single point, where the pendulum is at rest in its lowest position. Such a point is called a *point attractor* of the system. The role of attractors in a dynamical system will be discussed in more detail below.

2.15 Dissipative systems The damped pendulum is also an example of a *dissipative* system. Imagine a cluster of initial conditions, having a certain volume in phase space, and following this cluster over time. If the volume of the cluster decreases over time, the system is called dissipative. The criterion of dissipation is important because attractors are only present in dissipative systems.

The physical interpretation of a dissipative system is that a certain quantity in the system (usually its energy) is dissipated from the system. In the example of the pendulum, dissipation is caused by friction; friction decreases the speed of the pendulum, and thereby shrinks the range of possible speeds and the range of possible locations of the pendulum. When friction is removed (which is physically impossible), the system's behavior changes qualitatively. This situation is depicted in the rightmost frame of figure 1. Such a system is no longer dissipative and its trajectories become cycles.

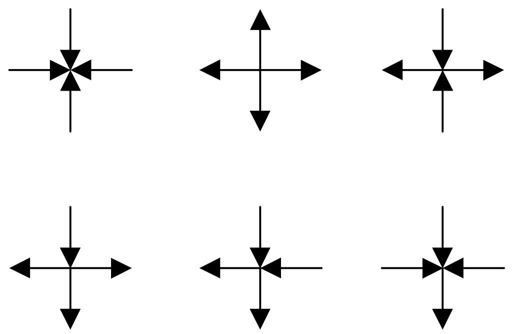

Figure 2: Examples of different kinds of fixed points.

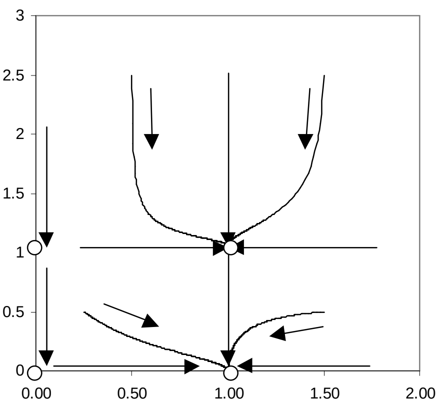

Figure 3: Example phase portrait of a simple dynamical system.

2.16 Bifurcation A change of the attractors due to a change of the control parameters of a system is called a *bifurcation*. The particular bifurcation of the pendulum system is not a very interesting one. As will be shown below, more interesting bifurcations take place in non-linear dynamical systems.

3. Non-linear dynamical systems

Linear dynamical systems do not have a very rich set of behaviors. Also, physical systems can not often be described exactly by a linear dynamical system. The linear systems that one often finds in physical models are often only approximations. The pendulum model that was used in the previous section is an example of such an approximation. Its validity is limited to pendulums with small angles. For larger angles the pendulum's behavior is considerably more complex. Most dynamical systems of interest are non-linear ones. As non-linear dynamical systems are generally not analytically solvable, their behavior can best be studied by calculation. Recently, with the availability of cheap computing power, interest in non-linear dynamical systems has increased enormously.

3.1 Phase portrait Much insight into the behavior of non-linear dynamical systems can be gained from studying their phase. The *phase portrait* of a system is a representation of all its trajectories. The complete set of trajectories is of course not useful to display, as there are an infinite number of trajectories. But, as was clear from the example of the pendulum, the qualitative behavior of the system is often more important than its exact behavior. It is therefore often quite sufficient to determine only the *fixed points* of the system and the behavior of the trajectories around these fixed points.
3.2 Fixed points *Fixed points* are points where the state variables do not change. This means that the derivative of each of the system's variables is zero at these points.
3.3 Stability Fixed points can be *stable*, meaning that nearby trajectories are attracted to the point, or *unstable*, so that nearby trajectories are repelled. Some fixed points are stable in one dimension and unstable in another.
3.4 Point attractor Fixed points that are stable from all directions are called *point attractors*.
3.5 Repellor Completely unstable fixed points are called *repellors*.
3.6 Saddle point Fixed points that are stable in one and unstable in another direction are called *saddle points*.
3.7 Basin of attraction The region over which trajectories are attracted towards an attractor is called its *basin of attraction*. Different kinds of 2-dimensional fixed points are schematically illustrated in figure 2.

Example 3.1: Phase portrait of a dynamical system

An example of a phase portrait of a simple dynamical system is given in figure 3. Its equations are:

$$\begin{aligned} \dot{x} &= x - x^3 \\ \dot{y} &= -\left(y^5 - 2y^3 + y\right) \end{aligned} \tag{4}$$

Note that x does not depend on y and y does not depend on x. This makes the analysis of the system very simple, but such independence does not usually occur in more realistic dynamical systems. Only the upper right quadrant of the phase space is shown, as the system is symmetrical around the axes. Fixed points are indicated with small circles. The point at (1,0) and that at (–1,0) is a true attractor. The other fixed points are all saddle points, because they attract in some directions and repel in others. The basin of attraction of the attractor at (1,0) is the region in phase space where $x > 0$ and $-1 < y < 1$. The basin of attraction of the saddle point at (1,1) is the region where $y \geq 1$ and $x > 0$.

Example 3.2: Other forms of attractors

Fixed points are not the only possible kind of attractor. Consider the dynamical system consisting of the following equations:

$$\begin{aligned} \dot{x} &= y + x\left(a - \left(x^2 + y^2\right)\right) \\ \dot{y} &= -x + y\left(a - \left(x^2 + y^2\right)\right) \end{aligned} \tag{5}$$

As illustrated in the left frame of figure 4, trajectories in this system converge towards an attractor point at (0, 0) for control parameter $a \leq 0$.

3.8 Limit cycle When a is strictly positive, the trajectories are no longer attracted to a point, but to a closed orbit, a so-called *limit cycle*. This is a prime example of a bifurcation where a point attractor changes into a limit cycle, a so-called Hopf-bifurcation.

4 Other attractors

Some examples of attractors have been shown. Intuitively, they are parts of the phase space towards which nearby trajectories converge. A technical definition for *attractors* is given by Strogatz, who defines an attractor as a closed set A with these properties:

1. *A is an invariant set*: any trajectory $\mathbf{x}(t)$ that starts in A stays in A for all time.
2. *A attracts an open set of initial conditions*: there is an open set U containing A such that if $\mathbf{x}(0) \in U$, the distance from $\mathbf{x}(t)$ to A tends to zero as $t \to \infty$. This means that A attracts all trajectories that start sufficiently close to it. The largest such U is called the *basin of attraction* of A.
3. *A is minimal*: there is no proper subset of A that satisfies conditions 1 and 2.

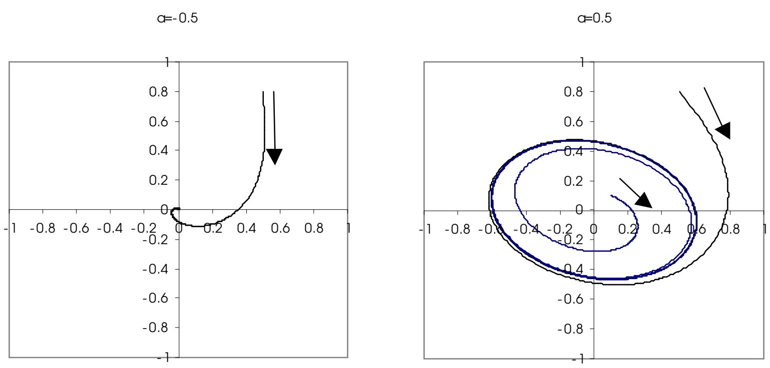

Figure 4: Example of bifurcation towards a limit cycle.

As trajectories cannot cross, the only kinds of attractors that can occur in two-dimensional nonlinear dynamical systems are point attractors and limit cycles. However in systems of three or more dimensions, other attractors are possible.

4.1 Strange attractor *Strange attractors* are attractors with a 'strange' geometry; a strange attractor is not periodic, but still contained in a finite volume of state space.

4.2 Fractal set Strange attractors can be *fractal sets*, i.e. sets that have a non-integer dimension.

4.3 Chaotic attractor If the trajectories on an attractor are sensitive to initial conditions, the attractor is called *chaotic*. Sensitivity to initial conditions means that the distance between two nearby trajectories increases exponentially over time:

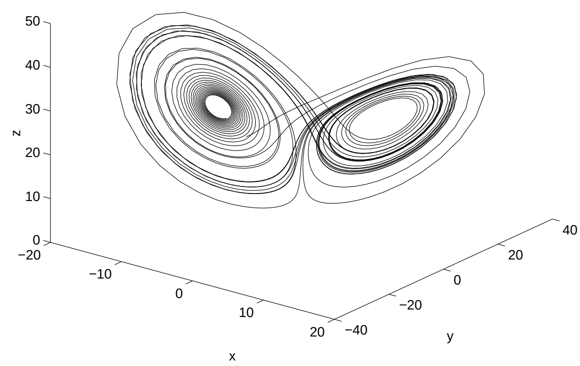

Figure 5: Lorenz attractor.

$$d(t) = d_0 e^{\lambda t} \tag{6}$$

(of course, as the attractor has finite volume, trajectories will eventually approach each other again).

4.4 Lyapunov exponent The exponent λ is called the *Lyapunov exponent*. The formal definition of sensitivity to initial conditions is that there is a Lyapunov exponent which, averaged over the initial conditions, is positive.

The terms *strange attractor* and *chaotic attractor* are not always used consistently in the literature. This confusion is probably due to the fact that many attractors are both strange *and* chaotic. However, both chaotic attractors that are not strange and strange attractors that are not chaotic exist.

5 Chaos

Sensitivity to initial conditions is an important property of chaotic systems, as it makes it impossible to predict their exact development on the long term. Strogatz defines chaos as: "*Aperiodic long-term behavior* in a *deterministic* system that exhibits *sensitive dependence on initial conditions*," where aperiodic long term behavior means that there are trajectories which do not settle down to non-chaotic attractors (fixed points, limit cycles, or quasiperiodic orbits) as $t \to \infty$. This condition is the same as that of a trajectory on a chaotic attractor.

In real physical systems it is impossible to measure or duplicate initial conditions with unlimited accuracy. An example of a physical chaotic system is the weather. The atmosphere is a chaotic system, and its initial conditions are impossible to measure exactly. Due to the chaotic nature of the weather, these inaccuracies are enlarged over time, and long-term weather predictions are therefore impossible to make.

It should be stressed that chaotic behavior is something quite different from random behavior, although they might look superficially the same to a casual observer. Chaotic systems are deterministic, but unpredictable because one would have to measure the state of the system with infinite accuracy. Stochastic systems are non-deterministic and therefore inherently unpredictable, even if one could in principle measure a system's state exactly.

Example 5.1: The Lorenz attractor

One of the first chaotic attractors that have been discovered was the Lorenz attractor. The equations for the Lorentz dynamical system are as follows:

$$\begin{cases}\dot{x} = a(y - x) \\ \dot{y} = -xz + bx - y \\ \dot{z} = xy - cz\end{cases} \tag{7}$$

with, for example, $a = 10$, $b = 28$, and $c = 8/3$. The resulting strange attractor (or in fact, a part of it) is shown in figure 5. This object is an attractor, as nearby trajectories are attracted towards it. However, the attractor is folded such that it never repeats itself. This folding is done in such a way that two points that are originally nearby will end up far apart after some time. This is an example of the sensitivity to initial conditions that is characteristic of chaotic systems. Sensitivity to initial conditions distinguishes chaotic systems from other aperiodic ones, such as:

$$x = \sin(t) + \sin(t\sqrt{2}) \tag{8}$$

that also do not repeat, but that do not show the same sensitivity to initial conditions.

Systems generally exhibit chaotic behavior only for a certain range of parameter settings, and the Lorenz system of the example is no exception. It shows bifurcations from point attractors to periodic attractors to chaotic attractors. The Lorenz system, however, is rather hard to study. A much simpler example is the discrete system called the *logistic map*, which will be described in the following example.

Example 5.2: The logistic map

The logistic map:

$$x_{t+1} = ax_t(1 - x_t) \tag{9}$$

displays interesting bifurcation behavior. This is illustrated in figure 6. As can be seen in this figure, the system converges to one attracting point for values of $a \leq 3$. After this, a limit cycle forms, consisting of two alternating values. The attracting point splits in two, and this explains the name bifurcation for such events. At $a \approx 3.45$, the system bifurcates into a cycle of length 4. The doubling continues with smaller and smaller intervals of a, until finally a truly chaotic stage is reached. Within the chaotic range, periodic ranges reappear (for instance, a window with period three is visible near $a = 3.83$). Something that is not very well visible in the figure is that the same kinds of bifurcations and transitions into chaos occur here as occurred in the original bifurcations. The bifurcation diagram is said to be *self-similar*. The property that enlargements of the parts resemble the original is common in chaotic systems. Geometric objects with this property are often *fractals*. Such patterns manifest themselves in many different ways in nature, for example in the shape of trees, clouds, mountains, and patterns of spots on certain shells.

As will be seen below, dynamical systems modeling can be helpful in understanding natural phenomena. So far, only mathematical models have been presented in any detail. As has been shown, terms such as attractor, bifurcation, chaos, etc. have precise meanings in dynamical systems theory. One should therefore be cautious when using the terminology of dynamical systems in connection with systems that have not been modeled mathematically. In the next section, examples of self-organizing complex dynamical systems will be presented.

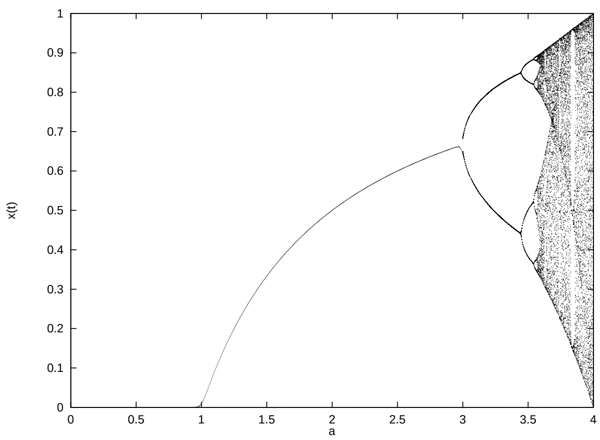

Figure 6: Bifurcation diagram of the logistic map.

6 Self-organization

6.1 Self-organization (emergence) The spontaneous emergence of order in a system is called *self-organization* or *emergence*. No generally accepted definition of self-organization exists. However, a principle common to most descriptions is that it refers to the development of order in dynamical systems, where this order is not a result of some central organizing force, but of (local) interactions between the elements of the system. There is no hierarchical organization of the system's parts. The system as a whole moves towards an attractor and due to the interactions between the system's parts, this attractor exhibits organization on a global scale. Self-organization can result in states that look very simple, but often the resulting states show marvelous complexity. Examples of this will be given below. Other properties that have been connected to self-organization are *coherent behavior* of a great number of elements, and *dissipative structures*.

6.1.1 Coherent behavior (Synergetics) It is clear that even quite simple dynamical systems are already capable of highly complex behavior. It would therefore seem that large dynamical systems with many degrees of freedom are too complex to be understood. However, if there is *coupling* between the different degrees of freedom in a system, it can start to behave as if it were a much simpler system with only a few degrees of freedom. In such a system there is slaving between the different components of the system.

The property of coherent behavior has its origin in the field of synergetics, a term coined by Haken. A good example is given by the fireflies that flash in unison in groups of thousands of individuals. By periodically flashing a light at a firefly, an experiment that was already proposed by Norbert Wiener in the heyday of cybernetics, it has been shown that the insect synchronizes its flashes for periods near its natural period of about 0.9 seconds. This type of order, resulting from interactions between the elements, explains the name self-organization.

Another typical example of synergetics is the laser. A laser consists of about 10^{27} molecules, which in normal circumstances would all emit light independently. However, when lasing takes place, all molecules will act in synchrony and the system can be described by only a few variables.

6.1.2 Dissipative structures Dissipative structures are seen as characteristic of self-organization in the school of Ilya Prigogine.

6.1.3 Entropy The second law of thermodynamics states that the entropy (degree of disorder) of a closed system can only increase, or remain constant, in which case the system has reached a state of equilibrium. This raises the question of how self-organization is possible, and how living systems can remain ordered (a condition for staying alive) instead of moving towards a state of thermodynamic equilibrium. The solution to this question is simple but interesting: living systems are not closed systems, but open systems. In open systems, there is a flux of entropy; entropy can enter or leave the system, in the form of energy or matter. In a version of the second law for open systems formulated by Prigogine, the *sum* of entropy production in the system (due to irreversible processes such as diffusion and chemical reactions) and entropy flux must be greater than zero. If the flux is negative, the total entropy of the system may decrease, providing a possibility for self-organization. To maintain this flux, dissipation is required. The term *dissipative structures* refers to structures that emerge as a result of self-organization and use dissipation to retain their organization.

6.2 Complexity If one tries to optimize a control parameter of a model in order to maximize a performance criterion (for example the number of cars per unit time in a traffic model), one sometimes finds that maximum performance is achieved for control parameter settings where the system is very near chaotic behavior.

6.3 The edge of chaos (Complexity) This near-chaotic state is sometimes referred to as the *edge of chaos* or *complexity*.

6.4 Self-organized criticality The tendency for systems to self-organize towards a critical state is called *self-organized criticality*. A critical state is a state where the individual degrees of freedom of the system keep each other in a more or less stable balance. A clear example is that of a sand pile to which sand is added. Eventually, a critical slope is reached where new sand will not increase the angle anymore, but fall off. When starting with a sand pile that is too steep, it will not collapse completely, but stop sliding when it arrives at a similar critical state, at which small perturbations may easily cause new collapses.

6.4.1 Catastrophe For self-organized criticality to occur, a system needs to be driven from the outside, for example through input of energy or mass. A property of systems that are critically self-organized is that they will regularly re-organize themselves in order to remain in the critical state. Such changes are called catastrophes.

6.4.2 Zipf's law The magnitude of these catastrophes can be described by the 1/*f*- or Zipf's law. This law states that the frequency of catastrophes is inversely proportional to their magnitude. Thus, catastrophes of all sizes occur. This has been related to a self-similar, fractal structure of the system that could be required to produce catastrophes of many different magnitudes. Some researchers claim that self-organized criticality is ubiquitous, but these claims seem to be rather exaggerated.

An interesting example of such a self-organized critical system has been investigated by Frette *et al*. of Oslo University, see figures 7 and 8.

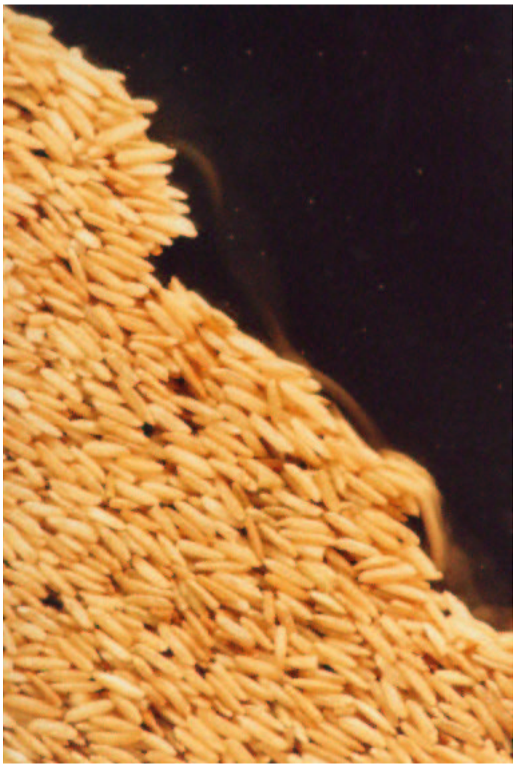

Figure 7: Self-organized critical rice pile (courtesy of Anders Malthe-Sørenssen).

7. Examples of self-organization and chaos

Self-organization can result in states that look very simple, but often the resulting states show marvelous complexity. In this section, examples of self-organization and chaos in real world systems will be described. These examples illustrate that self-organization and chaos occur in real-world systems from a variety of domains.

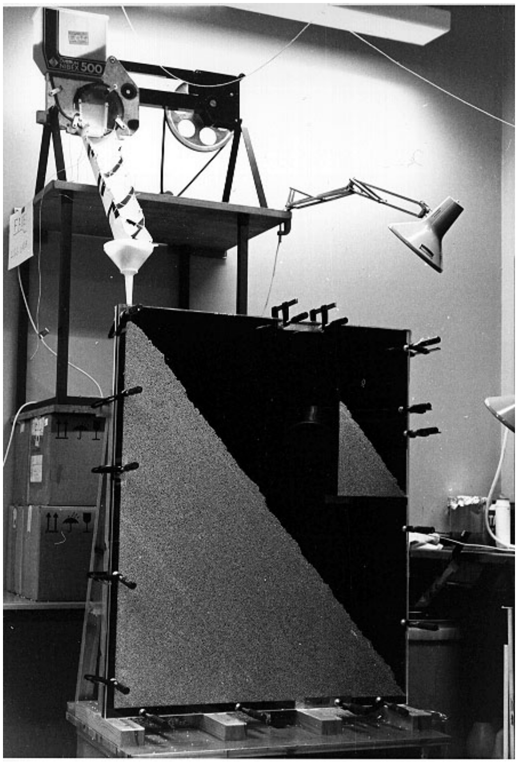

Figure 8: Experimental setup for investigating a self-organized rice-pile (courtesy Anders Malthe-Sørenssen).

7.1 The Belousov-Zhabotinski reaction

An example of a temporal dissipative structure is the Belousov-Zhabotinski reaction, often referred to in the context of self-organization, when it takes place in a well-stirred medium. In this case, the concentrations of Br^- and Ce^{4+} / Ce^{3+} ions are spatially homogeneous, but oscillate over time. The same reaction, when taking place in an unstirred shallow layer such as a petri dish, yields concentration waves with cylindrical symmetry or rotating spiral waves that are clearly visible with the naked eye.

Roux and colleagues showed that the Belousov-Zhabotinski reaction has strange attractors in the mathematical sense of the word. To demonstrate that the reaction dynamics are chaotic, they set up the reaction in a continuous flow stirred tank reactor. The rate with which fresh chemicals are pumped through the reactor (to keep the system far from equilibrium) is a control parameter in the experiment. When measured over time, the bromide concentration appears to be periodic, but is in fact aperiodic. If the system is to be chaotic, this aperiodicity should not be just random, but correspond to chaotic motion on a strange attractor. This was demonstrated in the following way. For systems governed by an attractor, the dynamics in the full phase space can be reconstructed from a time series of just a single variable; the method that allows this is called attractor reconstruction. Plotting combinations of the value of this variable at time t against its value at time $t+\tau$ and $t+2\tau$ (where τ is a suitably chosen constant) yields a three dimensional figure of orbits. The orbits appear to be confined to a lower dimensional subspace, like the orbits in the Lorenz attractor of figure 5. To test whether this was indeed the case, a 2-dimensional section of the attractor was made by taking the intersection of the trajectories with a plane. This is called a Poincaré section. To a good approximation, these points lie on a parameterizable one-dimensional curve. This demonstrates that the trajectories were indeed confined to an approximately two-dimensional sheet.

The data points on this curve correspond to successive intersections with the plane. When pairs of subsequent coordinates along this curve are plotted against each other, the result is a one-dimensional map $X_{n+1} = f(X_n)$. The shape of the map was that of a continuous injective curve, i.e. for each X_n, only a single value for X_{n+1} was measured. Thus, in the phase space determined by the three subsequent concentration measures, the aperiodic behavior of the system is deterministic, since each point in the space uniquely determines the next state of the system.

The largest Lyapunov exponent of this map was computed, and turned out to be significantly greater than zero (0.3±0.1), which demonstrates that the attractor is chaotic. Further properties of the system concerning period doubling and the order of periodic states that were encountered when varying the flow rate support the finding that the system is chaotic.

7.2 Rayleigh-Bénard convection

When a thin layer of liquid is heated from below, and when the difference in temperature between the bottom and the top of the liquid layer is small enough, the temperature will decrease linearly with the height. When the temperature difference between top and bottom exceeds a certain threshold, the liquid will move and form convection rolls. Liquid will be heated at the bottom so that its buoyancy increases. It will move to the top, where it will dissipate part of its heat. It will then become heavier and sink again. The behavior of such systems was studied experimentally by Bénard in 1900 and explained mathematically by Lord Rayleigh in 1916. Rayleigh-Bénard convection is interesting for the study of self-organization and complexity for two reasons. First of all, it is the theoretical basis for one of the most studied models in chaotic dynamic systems theory, the Lorenz model, which has been described above. Secondly, in liquid layers in which Rayleigh-Bénard convection is taking place, self-organization often takes place. The convection rolls split up into convection columns that form a hexagonal space-filling pattern. Hence a global pattern (a hexagonal tiling) emerges spontaneously from a homogenous starting situation.

7.3 Turing patterns

Alan Turing, the brilliant mathematician who laid the theoretical foundations for the modern computer, suggested that a combination of chemical reactions and diffusion is adequate to account for the main phenomena of morphogenesis.

By means of simulation (which, since computers were not commonly available yet, he carried out by hand), Turing demonstrated the theoretical possibility of the emergence of structure in initially quite homogeneous systems. Morphogens are substances that determine the development of a cell (e. g. a pigment cell). By viewing the chemical substances that form patterns as morphogens, the reaction systems Turing investigated may explain how a variety of patterns in nature develop, including the tentacle patterns on Hydra, whorled leaves, and the dappled patterns on animal skins. A pattern of particular interest for which Turing proved that the interaction between reactions and diffusion of two morphogens is sufficient to produce it, is that of stationary waves on a ring of tissue.

The peaks in this case are uniformly spaced round the ring, and their number can be approximated by dividing the 'chemical wave-length' of the system into the circumference of the ring.

It was not until 1990 that Turing's predictions were confirmed experimentally. The reaction necessarily involves positive feedback on an activator and inhibitory process. Furthermore, the inhibitor should diffuse much faster than the activator. One of the problems was that in aqueous solutions, molecules diffuse at similar rates, which presented difficulties with respect to the second constraint. The solution was to use a gel in which the diffusion rates of the substances differed.
A sustained standing Turing-type nonequilibrium chemical pattern was experimentally demonstrated. As predicted by Turing the pattern was characterized by an intrinsic wavelength, as opposed to for instance geometrical properties, which determine the structure of the convections in a Bénard cell. The patterns that are observed in these experiments typically have wavelengths between 0.1 and 0.5 mm; if similar processes are responsible for the formation of patterns on animal skins, they must take place at an early stage, and increase in size as the organism grows.

7.4 Self-organization in the brain

One of the most striking findings in neuroscience of the past decades has been the discovery of a columnar organization of cortex. The developmental process in the mammalian brain that results in ocular dominance columns and orientation columns can be viewed as a form of self-organization. The organization into columns, and the organization governing the arrangement of these columns, develops in reaction to visual input from the environment.

Ocular dominance columns, located in the visual cortex, respond to input from one eye. Orientation columns are also determined by the inputs to which they respond. Here however, the distinguishing factor is not the eye that is the source of the light, but the orientation of the shape formed by the light. All cells in V1 (an area within the primary visual cortex), except those in its fourth layer, are orientation sensitive. They do not respond to diffuse light, and only weakly to spots of light, but respond strongly when the activating stimulus is a line of the right orientation that falls within the most sensitive part of the cells’ receptive field.

The description of self-organization that was given earlier rules out forms of organization that are caused by some external organizing force. One might wonder whether the organization into ocular dominance and orientation columns should not be viewed as the result of the light stimuli from two eyes and with different orientations. However, although these stimuli are indeed required for the development of the columnar organization, these forces on their own do not explain why neurons corresponding to a particular eye or light orientation are grouped together. The activity of the cortex groups together similar inputs that were more or less homogeneously distributed when they entered the cortex; therefore, the process may be viewed as self-organization.

The organization into orientation columns has another interesting property. Not only are the cells in a single column related by virtue of their orientation selectivity, but so are neighboring columns. When V1 is viewed as a two dimensional grid of columns, there is a gradual change in the orientation represented by a column as columns along one dimension of the grid are considered. Along the other dimension, this orientation sensitivity is constant, but alternating ocular dominance is found in subsequent columns. This model of V1 is called the *ice-cube model* because of its geometrical shape. It should be realized however that this shape is an idealization of the structures that are actually found. Although the principle of the organization is widely accepted, the rows of the cortical grid are far from straight or parallel.

The development of ocular dominance columns has been modeled computationally. In models studied by Miller and colleagues, correlations of the activity of different neurons is the basis for the organization. It has been shown that in combination with these correlations, very general assumptions about neural connectivity suffice to satisfactorily account for the formation of both dominance and orientation columns. The organization of the columns themselves, making up the structure of the map as a whole has not yet been explained satisfactorily. The self-organizing feature maps developed by Kohonen, based on a competitive principle that tends to group similar input vectors together, yield very similar map structures, and might point in the right direction for a complete explanation, although a correspondence between brain structure and model structure is not obvious.

7.5 The honeycomb

One of the simplest and most elegant examples of self-organization in nature is that of the honeycomb. Although the individual worker bees are not globally controlled, a global structure emerges that is highly organized in space. This structure emerges from local interactions between the bees. Every bee independently builds a cell. It does this by hollowing out an increasingly large space in the wax that makes up a honeycomb. Whenever the bee feels that another bee is working on the other side of the wall of the cell, it stops removing wax in that direction. As all bees are of the same size, a very regular (although not perfectly regular) hexagonal surface-filling pattern of cells emerges. Similar structures appear in wasps' nests, see figure 9.

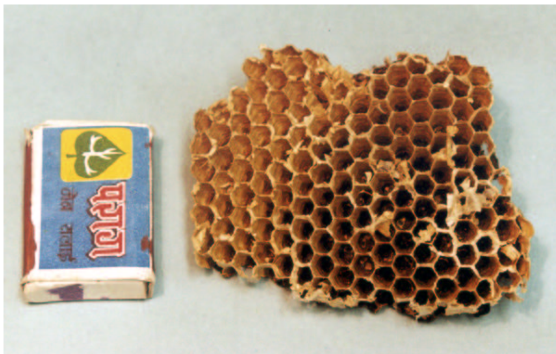

Figure 9: Wasps' nest with hexagonal pattern.

7.6 Traffic

In traffic control, global measures, such as the density of cars on a piece of road or the average flow, might seem to be sufficient for the purpose of measuring and controlling traffic. However, when the interactions between individual cars were studied, it turned out that these local interactions have effect on traffic flow that would otherwise not have been predicted. Moreover, the understanding gained from these individual-based models suggested a modification of traffic control that resulted in improved traffic flow when tested in a real traffic situation.

In a very early instance (1963) of individual-based modeling research that addressed traffic flow, Herman and Gardels developed a model based on individual driver-vehicle units. The behavior of a unit is by necessity a simplified representation of reality. It considers the car in front of a driver-vehicle unit as a stimulus that serves as input to the behavior of that unit. The behavior of the unit is represented by acceleration and braking, which limits the model to single lane traffic. Two parameters in the model are the sensitivity coefficient, which specifies the strength of responses to stimuli, and the response lag, which determines how long it takes before a driver reacts after detecting an event that calls for a reaction.

One of the issues that had to be further specified for building the model was the nature of the stimulus on which drivers act. When the parameters of the model were measured on a test track it was found that, rather than reacting to the distance to the next car, drivers react to the relative speed between the two cars. Furthermore, it was found that the sensitivity coefficient (the intensity of the response per unit stimulus) varied inversely with the spacing between cars.

Interesting findings resulted from measurements of actual traffic situation on three tunnels in New York. The model allowed the researchers to calculate the characteristic speed, i.e. the speed at which maximum flow occurs. Surprisingly, the estimation of 19 miles per hour, based on the measurements of two cars on a test track, corresponded very well to measurement of 24 000 cars in actual traffic (18.2 Mph).

There are two typical behaviors of an individual car: when traffic density is low, a driver selects a desired speed. In this situation, the throughput of traffic increases with traffic density, and this relation was identical for the three tunnels. However, at a certain point, interactions between cars start to play a role. These interactions cause disruptions and sometimes stoppages. The pattern of these disruptions is similar to a wave. Once some car in a flow of traffic slows down, the car behind it reacts and slows down, possibly more vigorously. In this manner, the event causes a wave of reactions that may eventually lead to stoppages, even though the car that originally caused the disruption may already have a large distance to the cars that are affected by this process.

The realization that such shock waves have major effects on traffic flow leads to the idea of introducing gaps in traffic, the idea being that shock waves are absorbed by these gaps and prevent the disruptions from spreading to upcoming blocks of traffic. The effects of this change were dramatic. Apart from an increase in hourly flow from 1176 to 1248 vehicles, congestion on tunnel approaches diminished because the flow through tunnels had increased. Furthermore, vehicle breakdowns were reduced by 25 percent, and the reduction of accelerations relieved tunnel ventilation requirements. The benefits of the changes were such that it was decided to implement the traffic control scheme permanently.

8 A software simulation environment: Swarm

One of the purposes of this article is to bring the possibilities for and possible benefits of modeling interactions among elements of a system under the attention of researchers from a variety of disciplines. As the article has demonstrated, this approach may capture essential properties of many different types of systems, and help explain the processes that occur in those systems that would be difficult or impossible to explain otherwise. Given this purpose, an important question is how researchers from these different disciplines are to become familiar with the tools necessary to undertake such an investigation. In particular, it may not be expected that the researcher has advanced skills in software design. For this reason, it has been considered useful to suggest a tool for creating individual-based simulations in software.

The tool suggested here is called Swarm, and has been developed by researchers at the Santa Fe Institute. Using Swarm does not take away the need for basic programming skills. Rather, its benefit is that the burden of developing a well-designed framework for simulation is taken off the shoulders of the researcher, who has often not been educated to perform this task. Another goal behind the development of Swarm was to provide a common language for describing simulation experiments, so that experiments reported in scientific papers may be repeated by other researchers.

The basic structures in a Swarm system are Model swarms and Observer swarms. A model swarm defines the world that is simulated and the elements that play a role in it. Furthermore, it contains a schedule specifying how the activities of the objects within a model swarm take place over time. An observer swarm contains structures that monitor the events in a model swarm, and a scheduler that determines the schedule of measurements. Observer swarms typically produce graphs or collect statistics of the model swarm.

Swarm has been used by researchers from a variety of disciplines including biology, ecology, anthropology, economics, political science, and geography. All information concerning Swarm, including the software itself, can be found at http://www.swarm.org. The Swarm libraries are distributed under the GNU Library Public License.

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