

Hierarchical-Alphabet Automata and Applications

Introduction

- **Who am I?**
 - PhD student at the AI Lab (Applied Research)
 - Under supervision of prof. Johan Loeckx
 - Research interests in applied AI and cybersecurity
- **What is this guest lecture about?**
 - Variants of finite state machines and applications
 - Our hierarchical extension on finite state machines

Outline of the lecture

- **Preliminary knowledge:**
 - Terminology + basics of automata theory
 - Tries, factor automata, and factor oracles
- **Our research at the AI lab:**
 - Hierarchical-alphabet automata (HAAs)
 - Hierarchical factor oracles (HFOs)
- **Applications of our research:**
 - Anomaly detection with the HFO

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Terminology

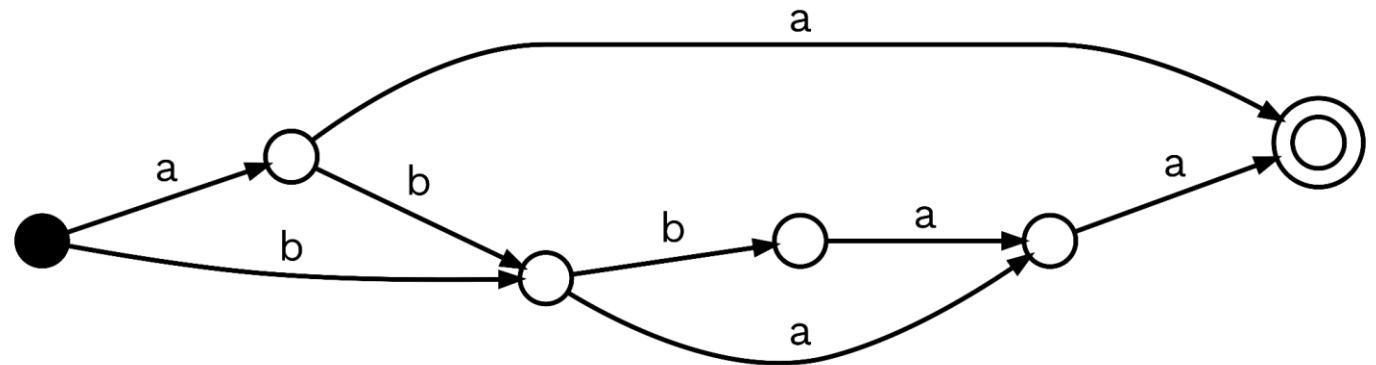
- **Alphabet:** finite set of symbols
- **Words:** concatenations of zero or more symbols
- **Factors:** contiguous subsequences of a word
 - **Prefix:** a factor at the beginning of a word
 - **Suffix:** a factor at the end of a word
- **Language:** subset of the infinite set of possible words we can create using an alphabet!

Terminology: Examples

- **Alphabet:** { a, b, c, ..., 3, 4 }
- **Words:** { "", "a", "abc", "cars", "3x4mpl3", ... }
- **Factors:** actor is a factor of factory
 - **Prefix:** fact is a factor and prefix of factory
 - **Suffix:** ory is a factor and suffix of factory
- **Languages:** a^*b^* , the set of factors of a word, ...

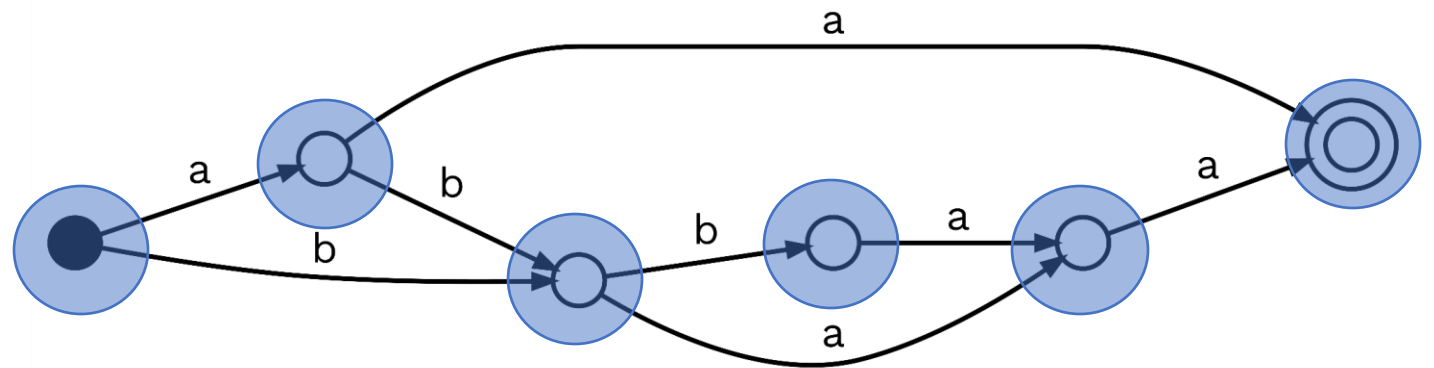
Finite State Machines

- **Definition:** a five-tuple $(Q, \Sigma, \delta, q_0, F)$
 - A finite set of states Q
 - An alphabet (or set of symbols) Σ
 - A transition function δ , returns a q of Q
 - The initial state q_0
 - A set of accepting states F



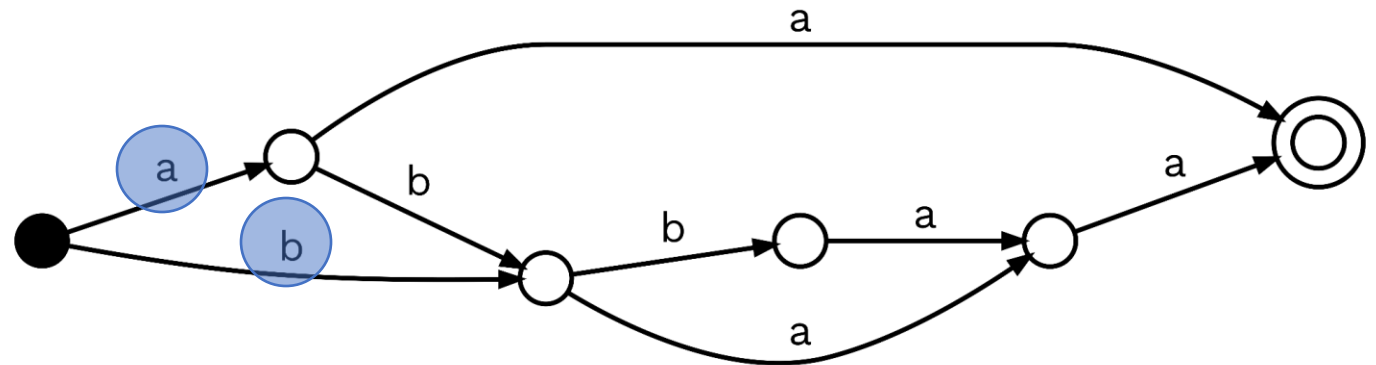
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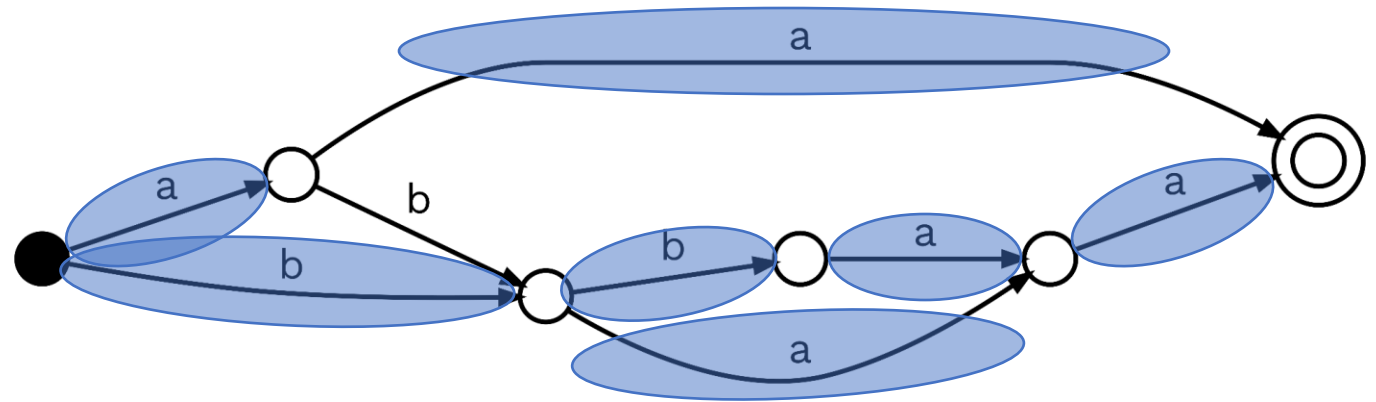
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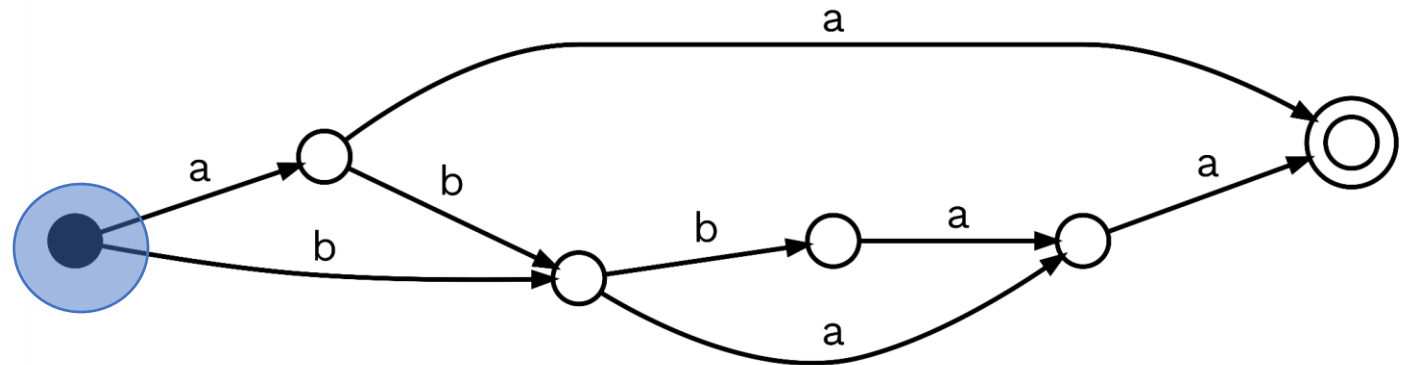
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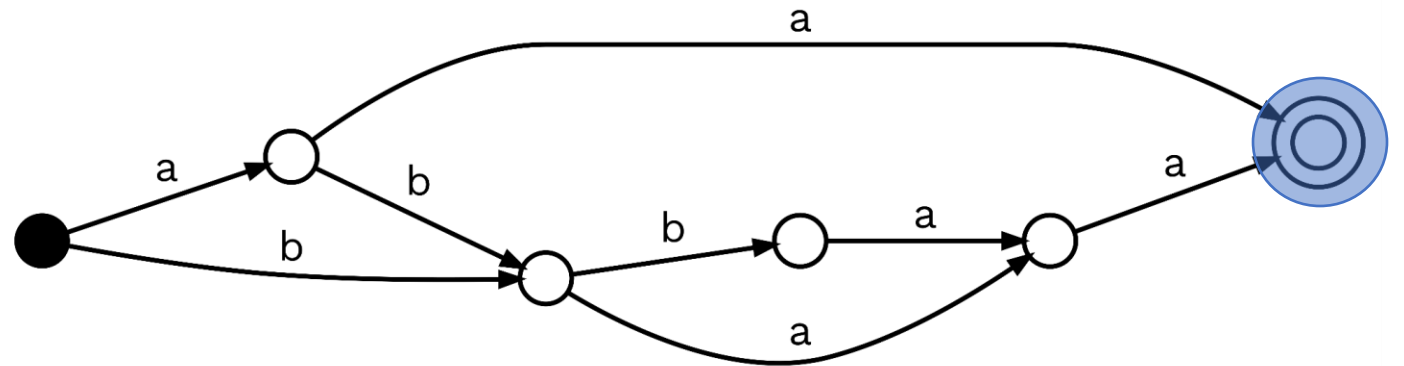
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Finite State Machines

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 - The initial state q_0
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Finite State Machines

- **Acceptance and rejection of words:**
 - A finite state machine recognizes a language:
 - It **accepts** words part of that language
 - It **rejects** words not part of the language
 - Start at the initial state q_0 + first symbol of your input word
 - Repeatedly follow δ with current state + symbol of word:
 - Accepting state after full word? Word is **accepted**
 - Normal state or no transition? Word is **rejected**

Applications of FSMs

- **Anomaly detection on system call sequences**

```

1. S0;
2. while (..) {
3.   S1;
4.   if (...) S2;
5.   else S3;
6.   if (S4) ... ;
7.   else S2;
8.   S5;
9. }
10. S3;
11. S4;

```

$S_0 S_1 S_2$	$S_1 S_2 S_4$	$S_2 S_4 S_5$	$S_3 S_4 S_5$	$S_4 S_5 S_1$	$S_2 S_5 S_1$	$S_5 S_1 S_2$
$S_0 S_1 S_3$	$S_1 S_3 S_4$	$S_2 S_4 S_2$	$S_3 S_4 S_2$	$S_4 S_5 S_3$	$S_2 S_5 S_3$	$S_5 S_1 S_3$
$S_0 S_3 S_4$				$S_4 S_2 S_5$		$S_5 S_3 S_4$

Figure 1. An example program and associated trigrams. S_0, \dots, S_5 denote system calls.

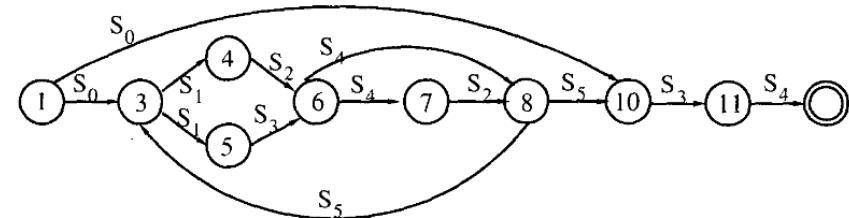
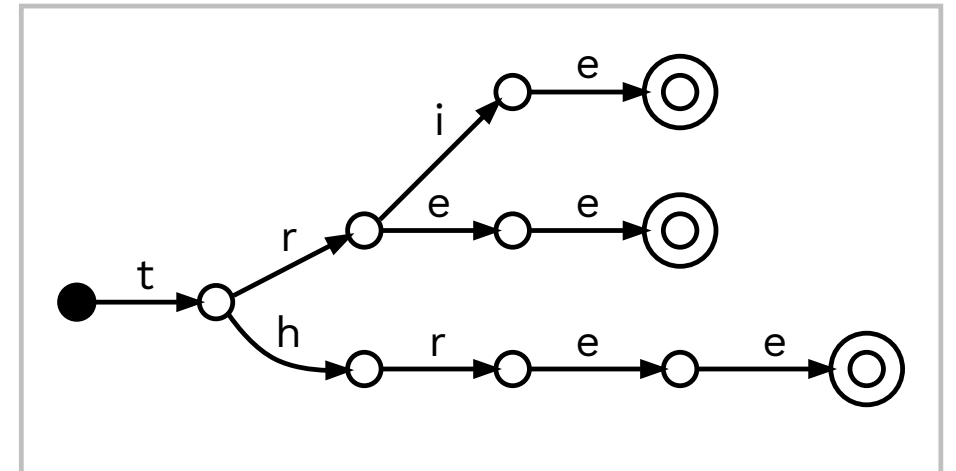


Figure 2. Automaton learnt by our algorithm for Example 1

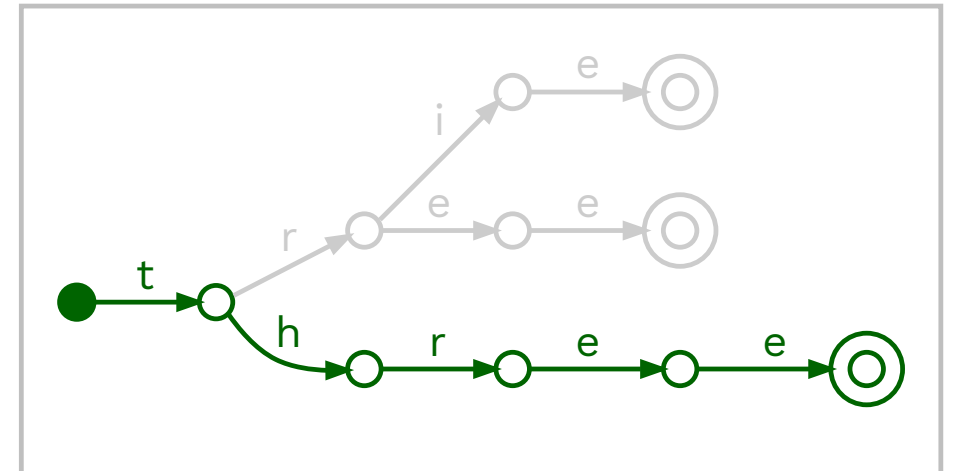
Tries

- **Definition:**
 - Rooted tree associated with a set of words
- **Paths from root to leaf represents words from its set**
- **Example:**
 - Trie for the set { *three, tree, trie* }



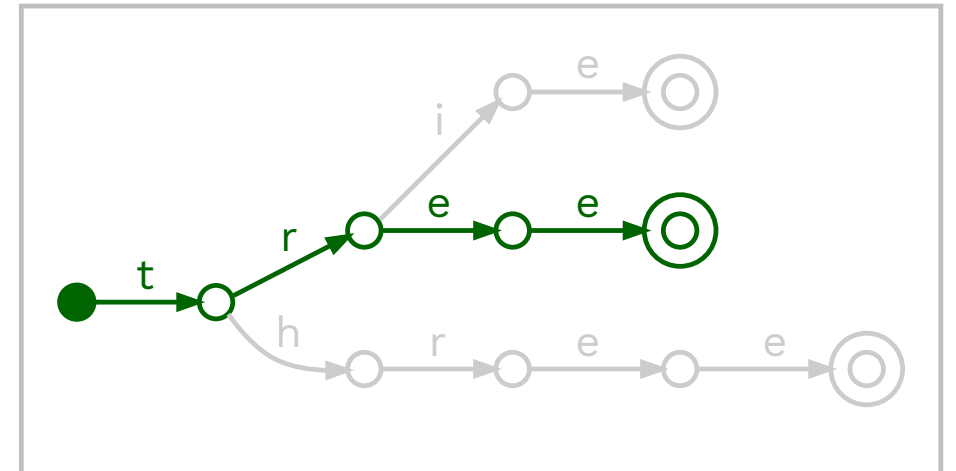
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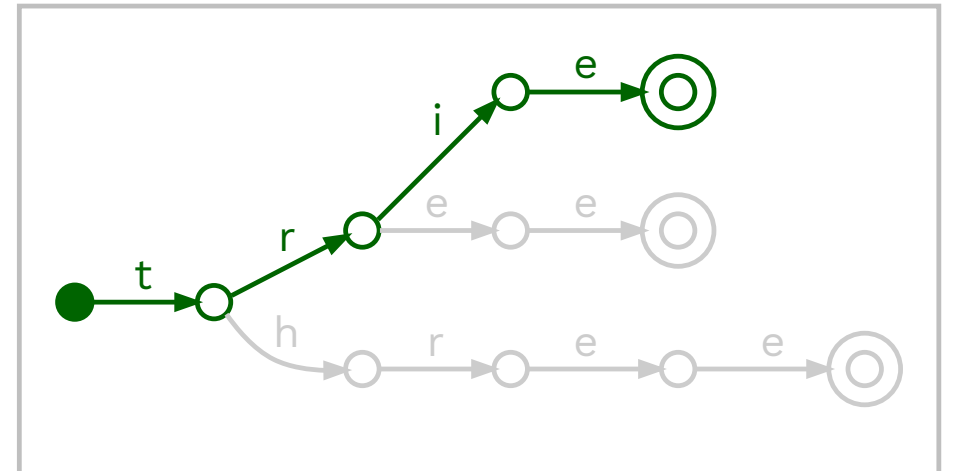
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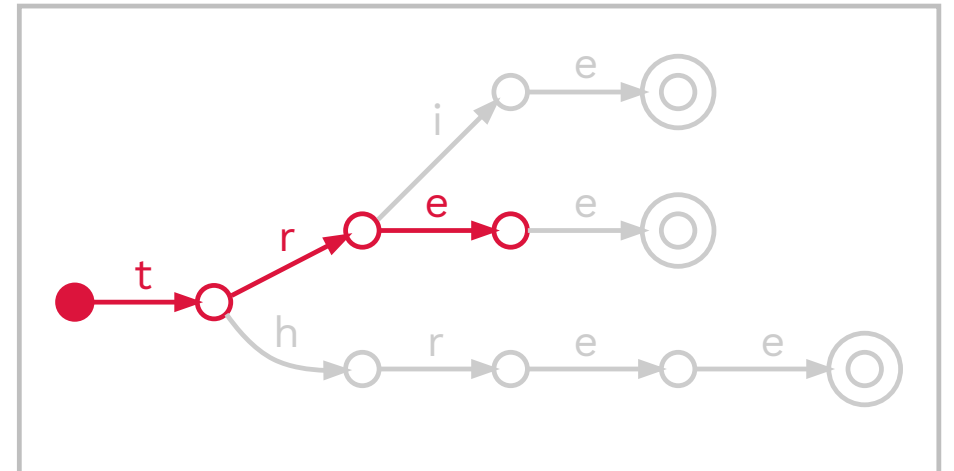
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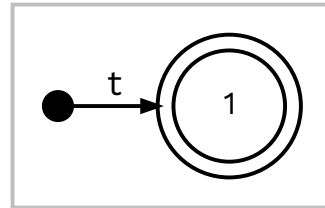


Tries

- **Definition:**
 - Rooted tree associated with a set of words
- **Paths from root to leaf represents words from its set**
- **Example:**
 - The word ***tre*** is not in { *three*, *tree*, *trie* }

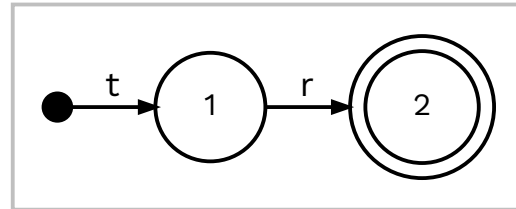


Construction Algorithm of Tries



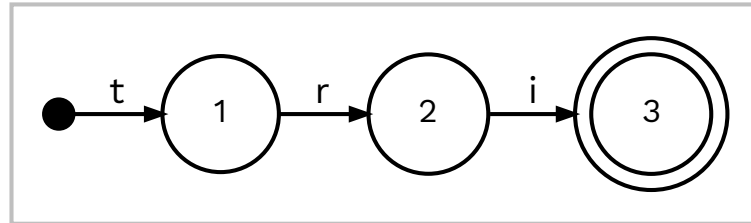
{**t**rie, tree}

Construction Algorithm of Tries



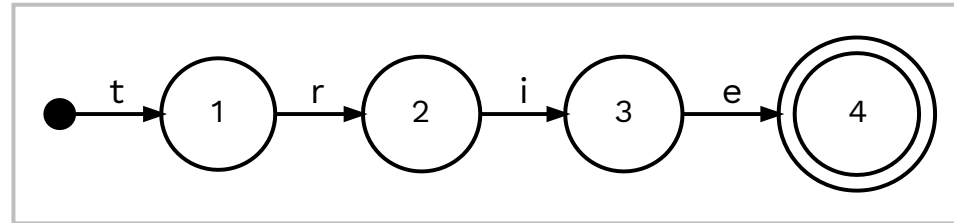
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Construction Algorithm of Tries



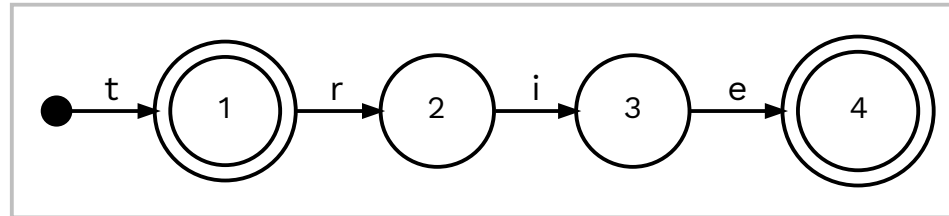
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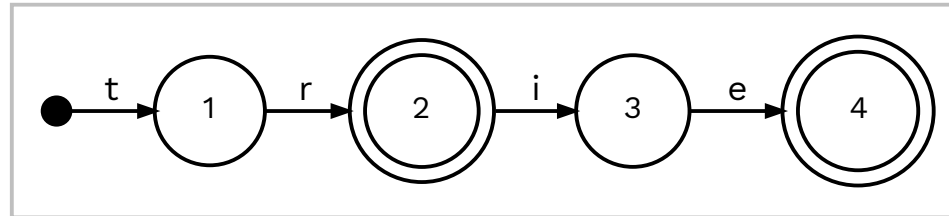
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Construction Algorithm of Tries



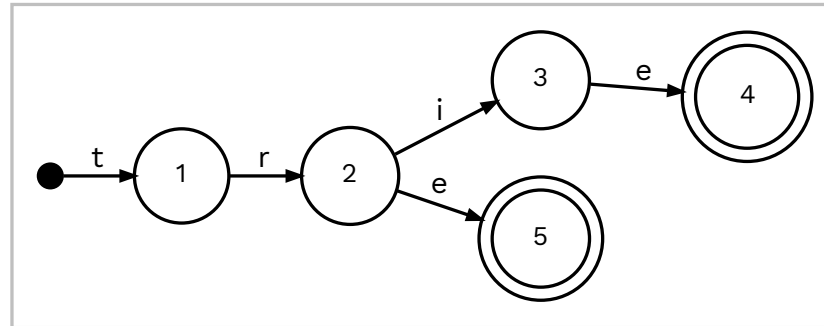
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Construction Algorithm of Tries



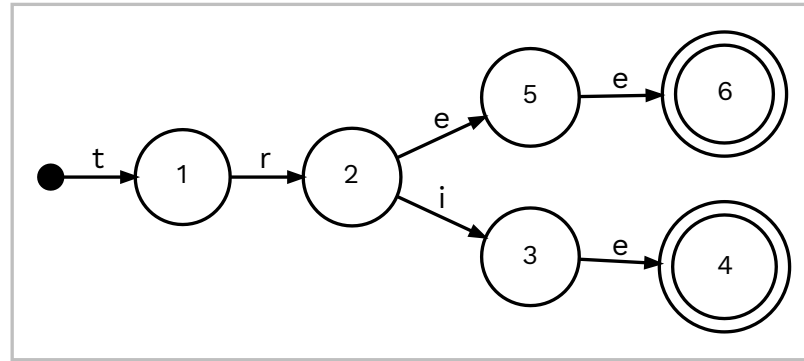
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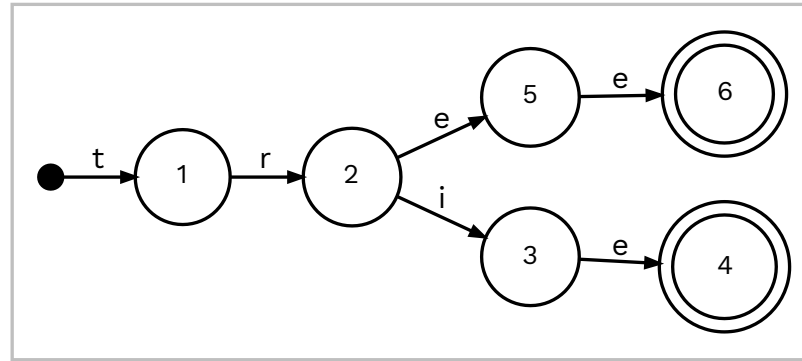
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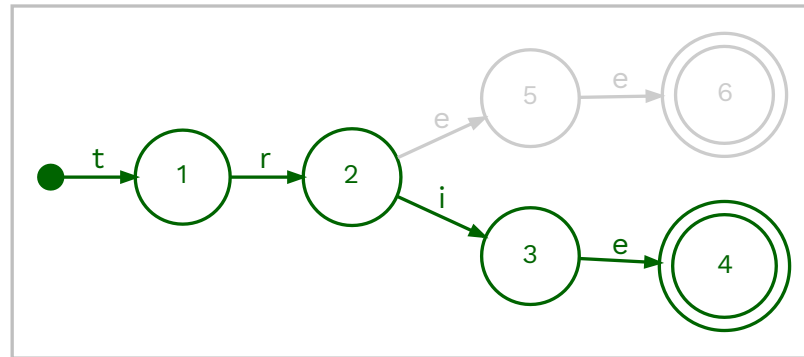
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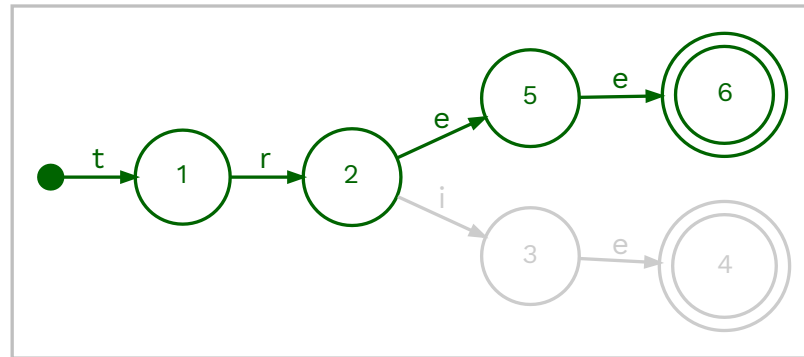
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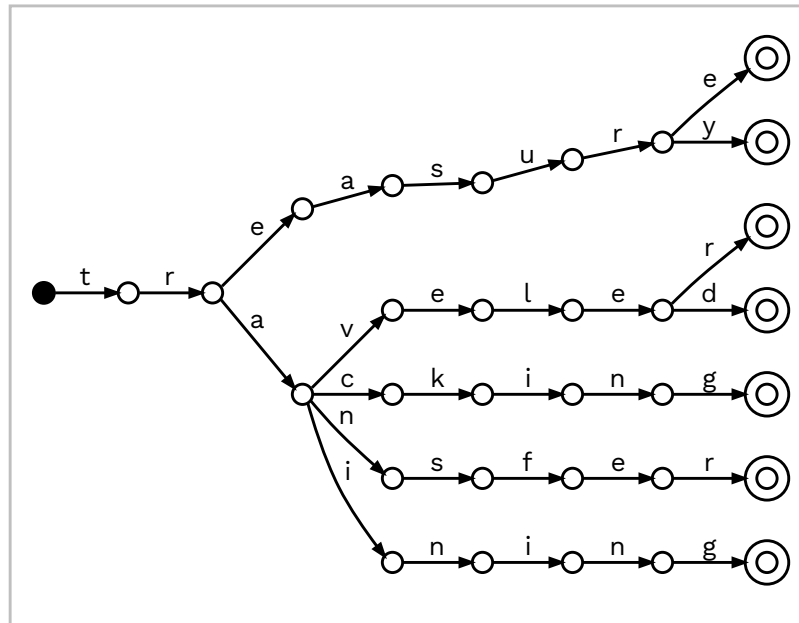
Construction Algorithm of Tries



{trie, tree}

Applications of Tries

- **Efficient data structure, storing words based on prefixes**
 - Eliminates duplicate prefixes, allows for efficient search



Interactive Demonstration of Tries

Factor Automata

- **Definition:**
 - Factor automaton of a word x is the minimal deterministic automaton that recognizes the factors of x

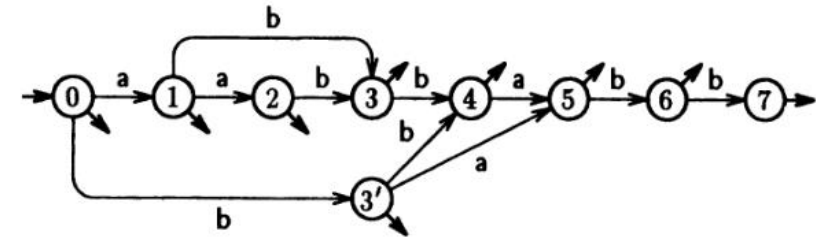
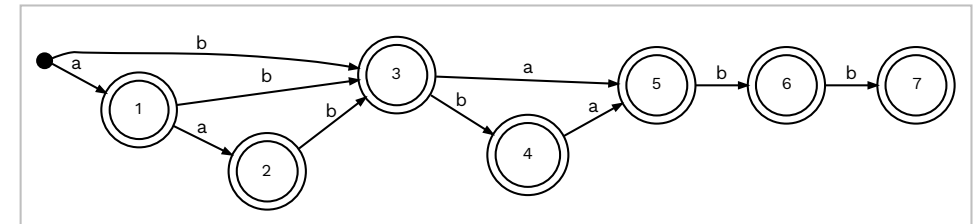


Fig. 7.11. Minimal deterministic automaton recognizing the factors of aabbabb.

Factor Oracles

- **Definition:**
 - Acyclic automaton built on a (set of) word(s), recognizes **at least** all factors of this (set of) word(s)
- **Applications:**
 - Intended for pattern matching
 - Computation of repeat factors
 - Modelling for music improvisation



Factor Automata vs Factor Oracles

- **Factor automaton:**
 - Deterministic automaton that recognizes the factors of x

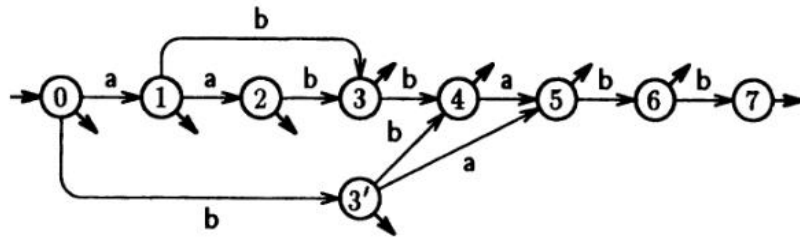
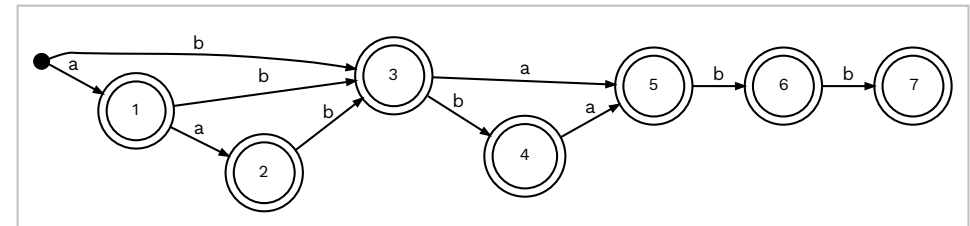


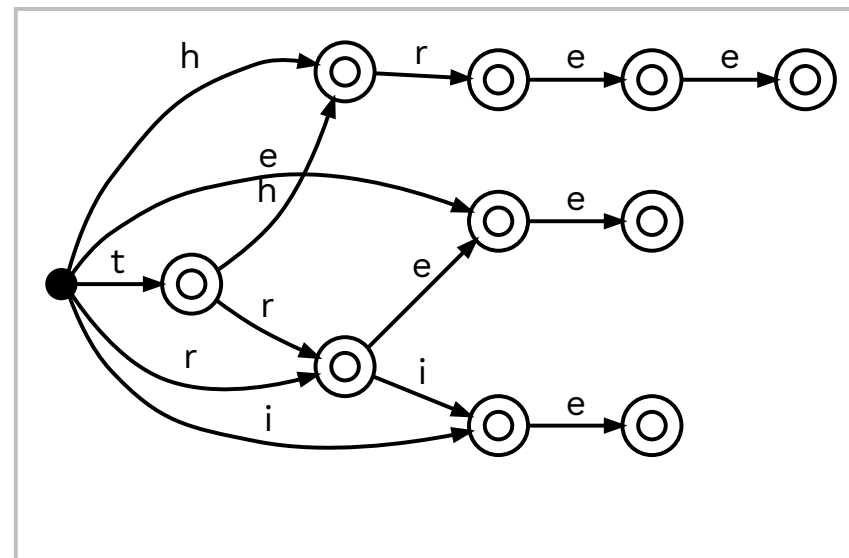
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- **Factor oracle:**
 - Acyclic automaton that recognizes at least all factors of x



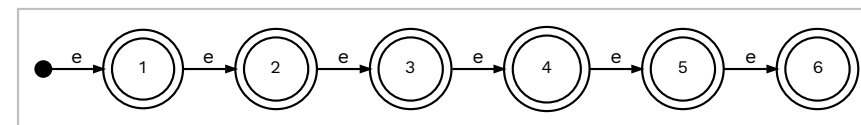
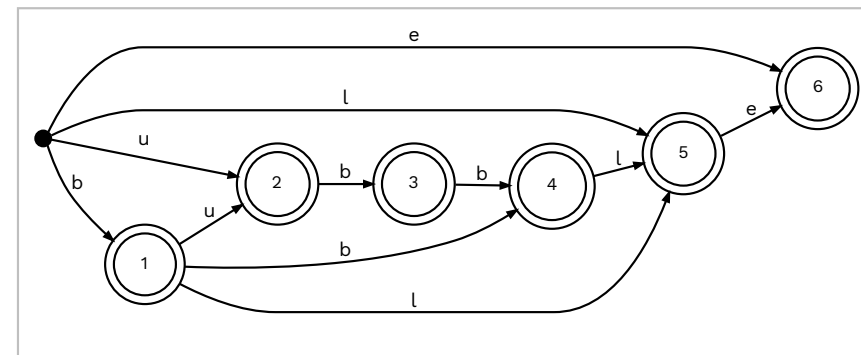
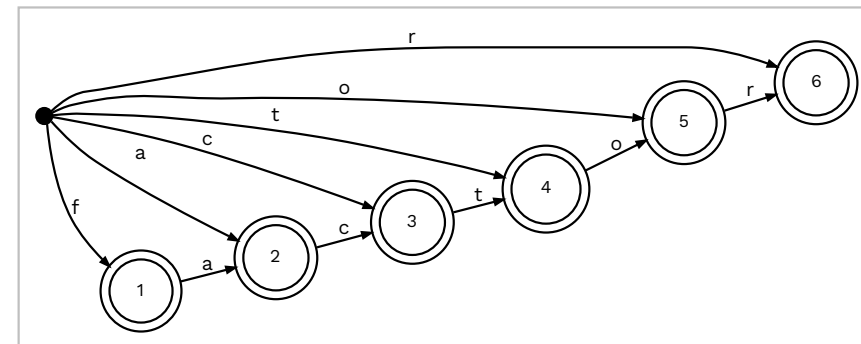
Factor Automata vs Factor Oracles

- **Factor oracle:**
 - Has an online algorithm
 - Constructed in linear time and space
 - Memory-efficient improvement over FAs
- **Example:**
 - Factor oracle for {three, trie, tree}



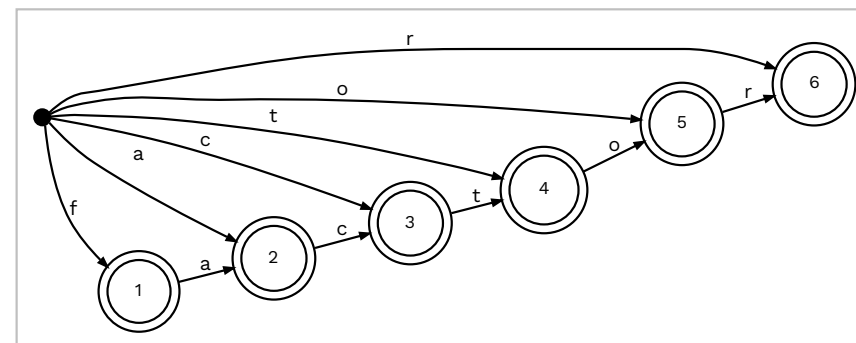
Factor Oracles

- **States and transitions:**
 - Has exactly $m + 1$ states
 - Between m and $2m - 1$ transitions
- **Examples:**
 - FO for the word "factor"
 - FO for the word "bubble"
 - FO for the word "eeeeee"



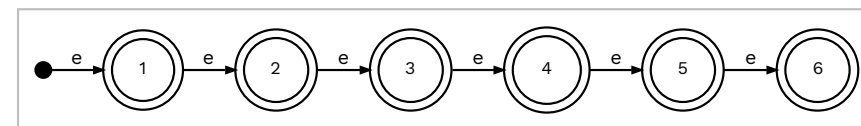
Factor Oracles

- **States and transitions:**
 - Has exactly $m + 1$ states
 - Between m and $2m - 1$ transitions
- **Example: "factor"**
 - Word has length 6
 - Has 7 ($= 6 + 1$) states
 - Has 11 ($= 2 * 6 - 1$) transitions
 - Why? No repeat symbols = **worst-case**



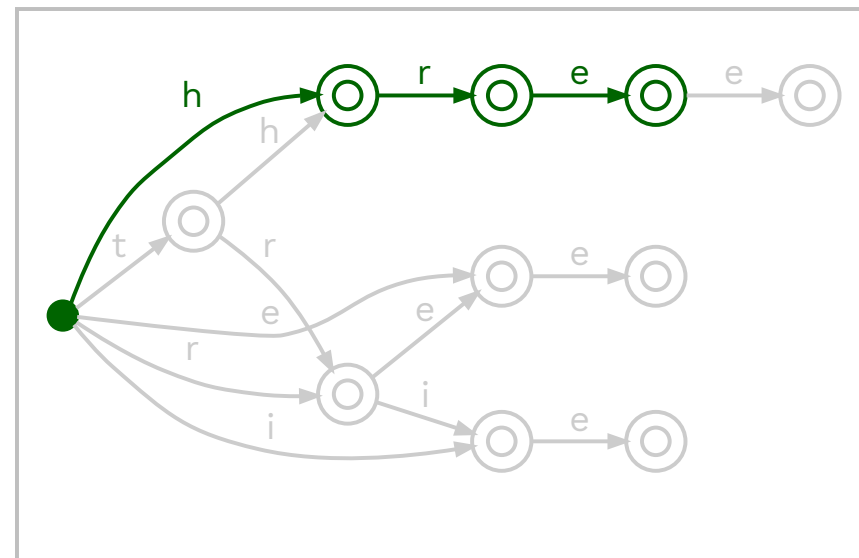
Factor Oracles

- **States and transitions:**
 - Has exactly $m + 1$ states
 - Between m and $2m - 1$ transitions
- **Example: "eeeeee"**
 - Word has length 6
 - Has 7 (= 6 + 1) states
 - Has 6 transitions
 - Why? Every symbol is equal = **best-case**



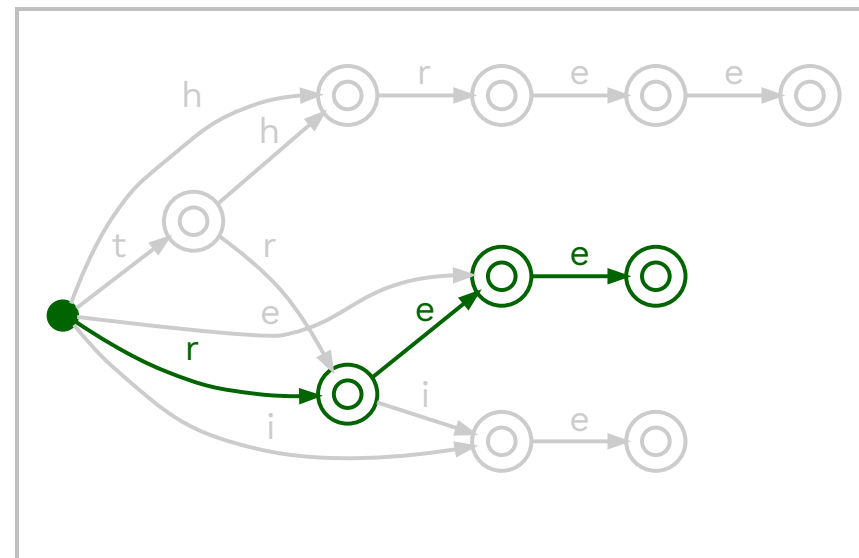
Factor Oracles

- **Accepts at least all factors:**
 - FO for {three, trie, tree}
- **Example:**
 - Accepts "hre", which is a factor of three



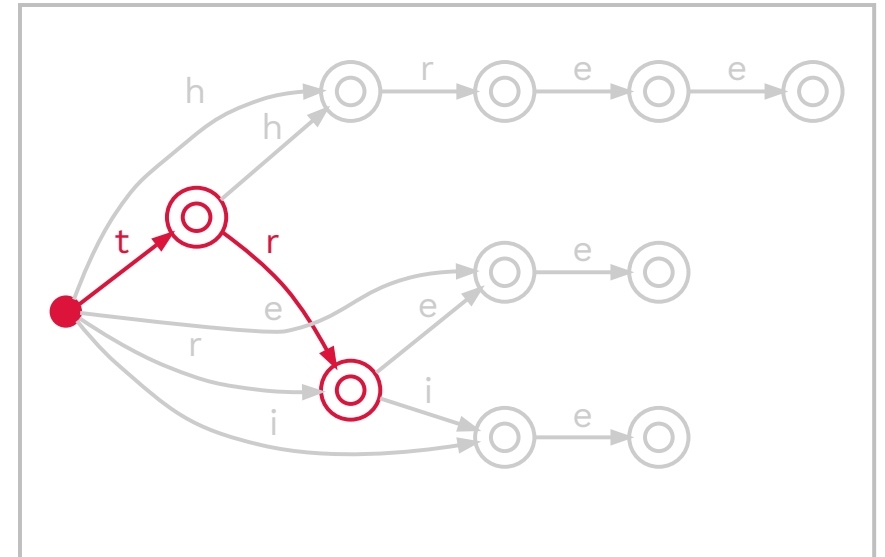
Factor Oracles

- **Accepts at least all factors:**
 - FO built on {three, trie, tree}
- **Example:**
 - Accepts "ree", which is a factor of both three and tree

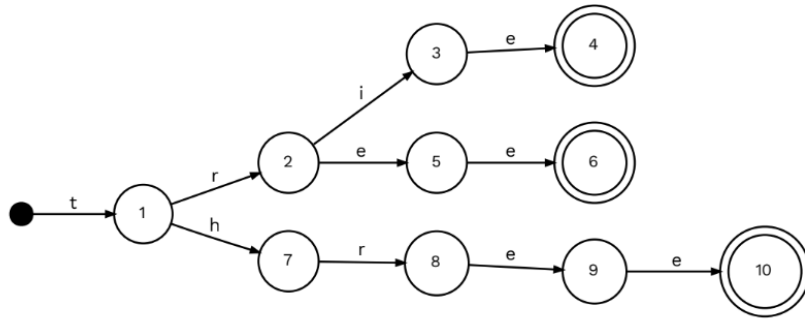


Factor Oracles

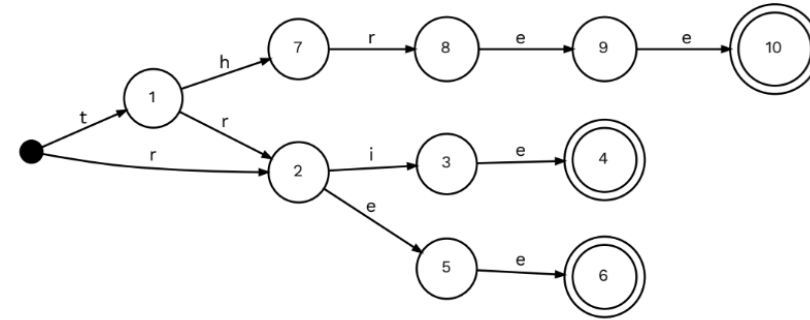
- **Accepts at least all factors:**
 - FO built on {three, trie, tree}
- **Example:**
 - Rejects "trh", because it is not a factor of three, trie, or tree



Construction Algorithm of FOs

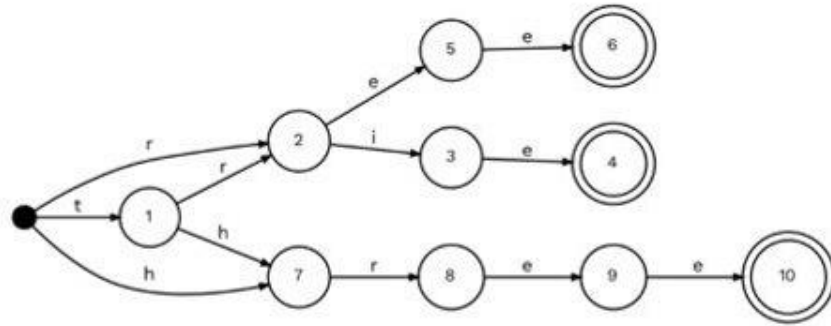


(a) The trie for $P = \{trie, tree, three\}$, taken from *Fig. 2.10*. In the first step of the algorithm, we generate the trie which we use as a starting point to generate the factor oracle.

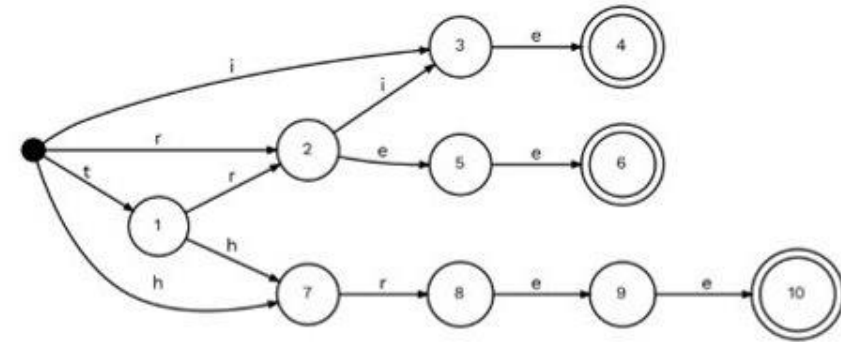


(b) In this step of the construction algorithm, a new transition **from the initial state to state 2** is made, labelled with r . There are now 11 transitions in total.

Construction Algorithm of FOs

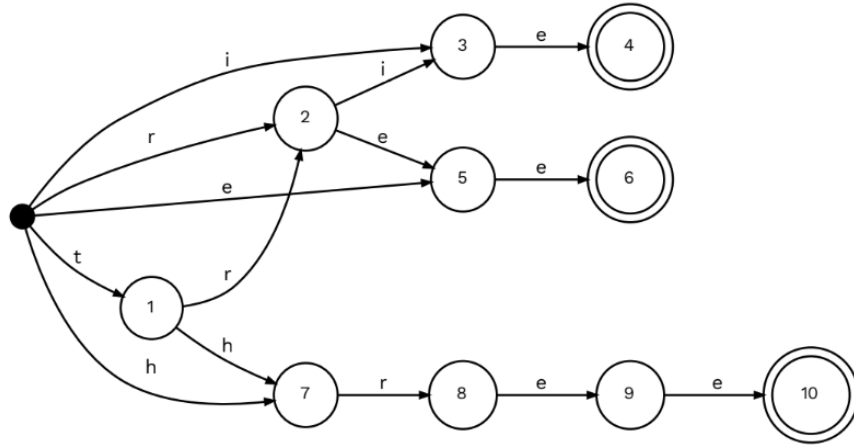


(c) In this step of the construction algorithm, a new transition **from the initial state to state 7** is made, labelled with *h*. There are now 12 transitions in total.

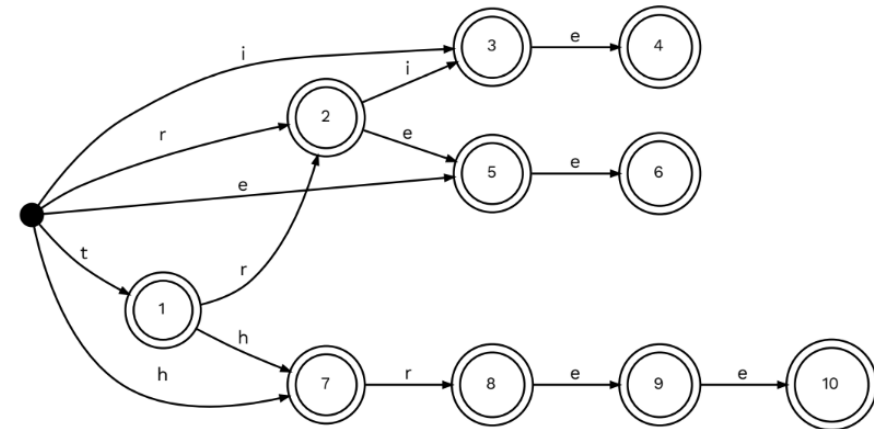


(d) In this step of the construction algorithm, a new transition **from the initial state to state 3** is made, labelled with *i*. There are now 13 transitions in total.

Construction Algorithm of FOs



(e) In this step of the construction algorithm, a new transition **from the initial state to state 5** is made, labelled with e . There are now 14 transitions in total.



(f) In the final step of the construction algorithm, **all states are marked as final states**. The algorithm transformed the original trie to the factor oracle of $P = \{trie, tree, three\}$.

Interactive Demonstration of FOs

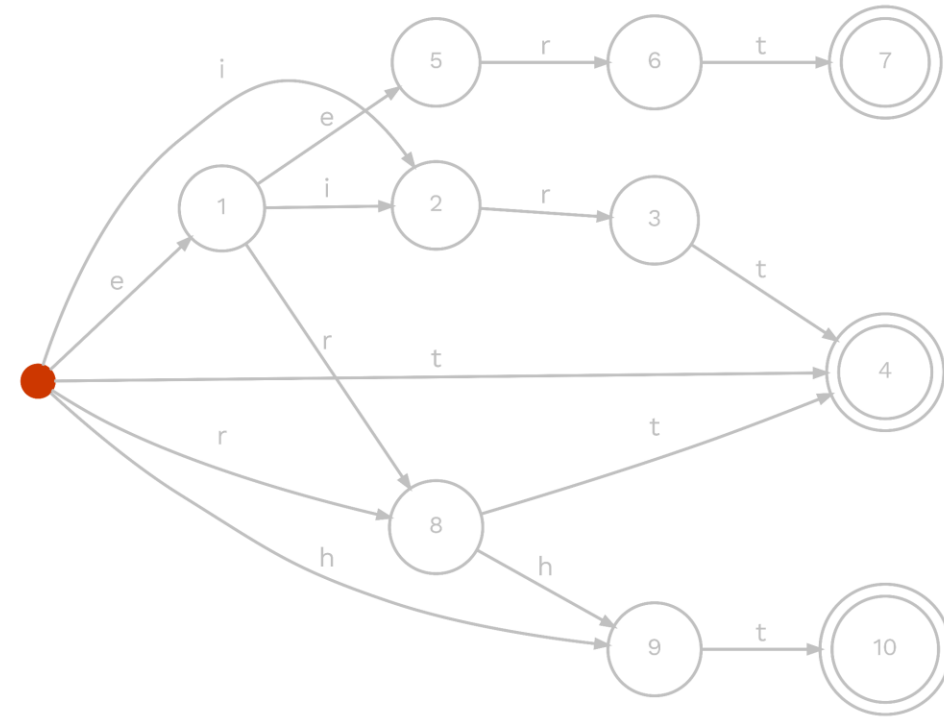
Applications of FOs

- **Used for pattern matching:**
 - Backwards Oracle Matching (BOM)
 - Set Backwards Oracle Matching (SBOM)

Set Backward Oracle Matching

`two_or_three_trees`

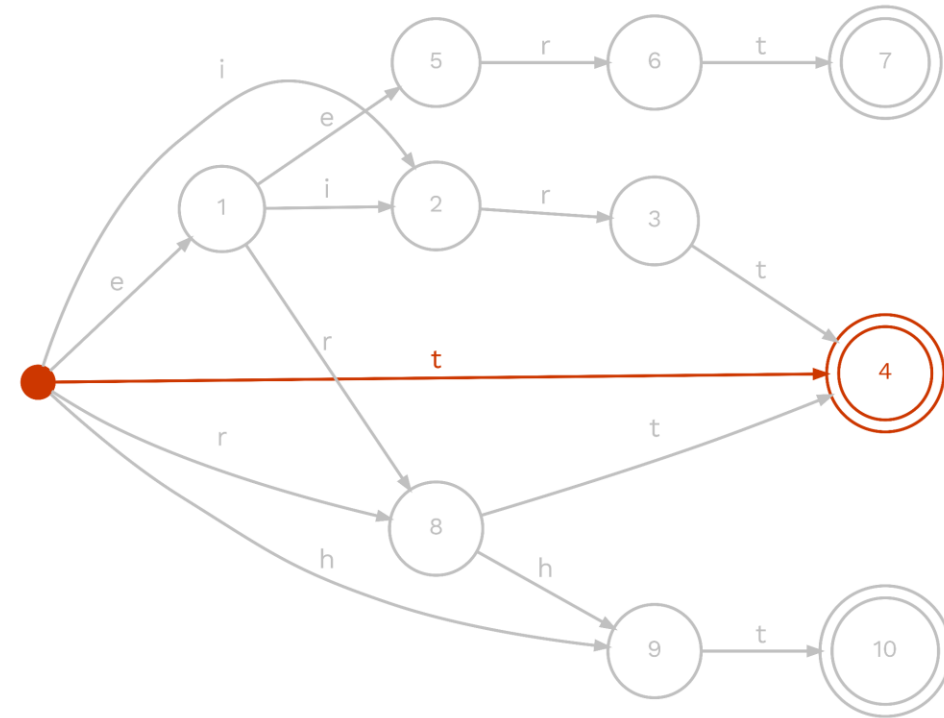
(a) We attempted to input `_` in the factor oracle, but the oracle rejects directly. We shift the window after the `_` and continue our search.



Set Backward Oracle Matching

two_or_three_trees

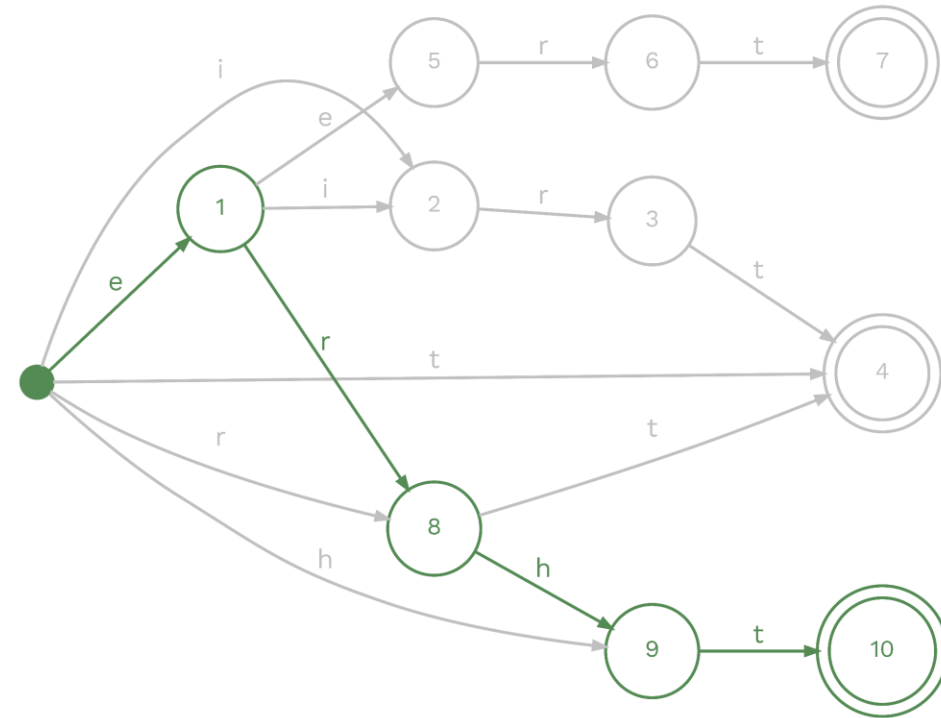
(b) We successfully input t in the factor oracle, but the oracle fails on $_$. We shift the window after the $_$ and continue our search.



Set Backward Oracle Matching

two_or_ **three** *e_trees*

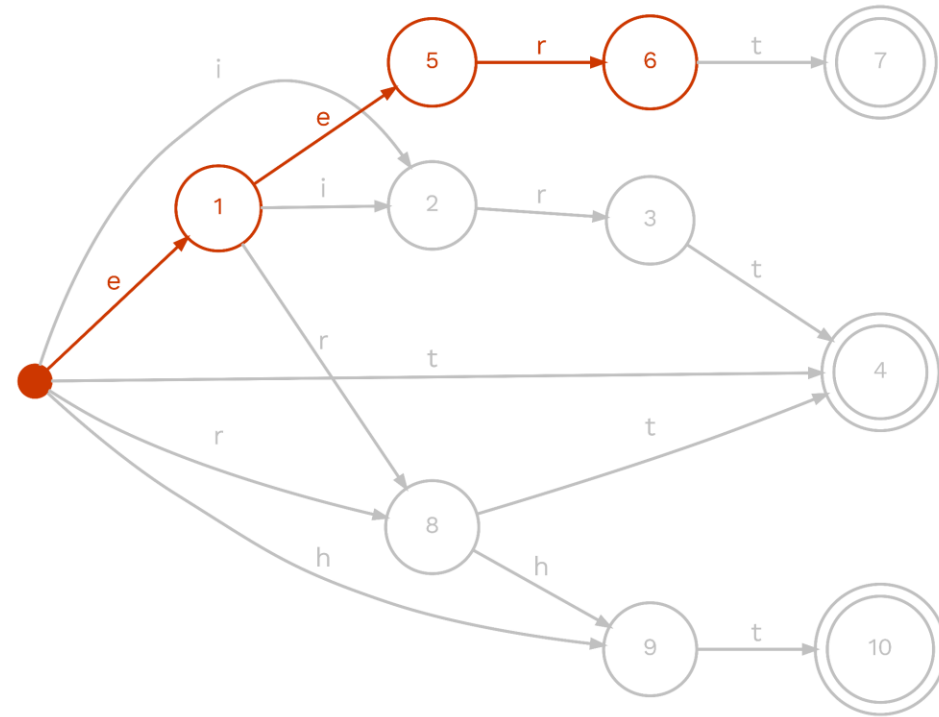
(c) We successfully input *e*, *r*, *h*, and *t* in the oracle. We reach accepting state 10, which is associated with the word *three*. We check for an occurrence of *three*, and shift the window by 1.



Set Backward Oracle Matching

*two_or_****t****hree_trees*

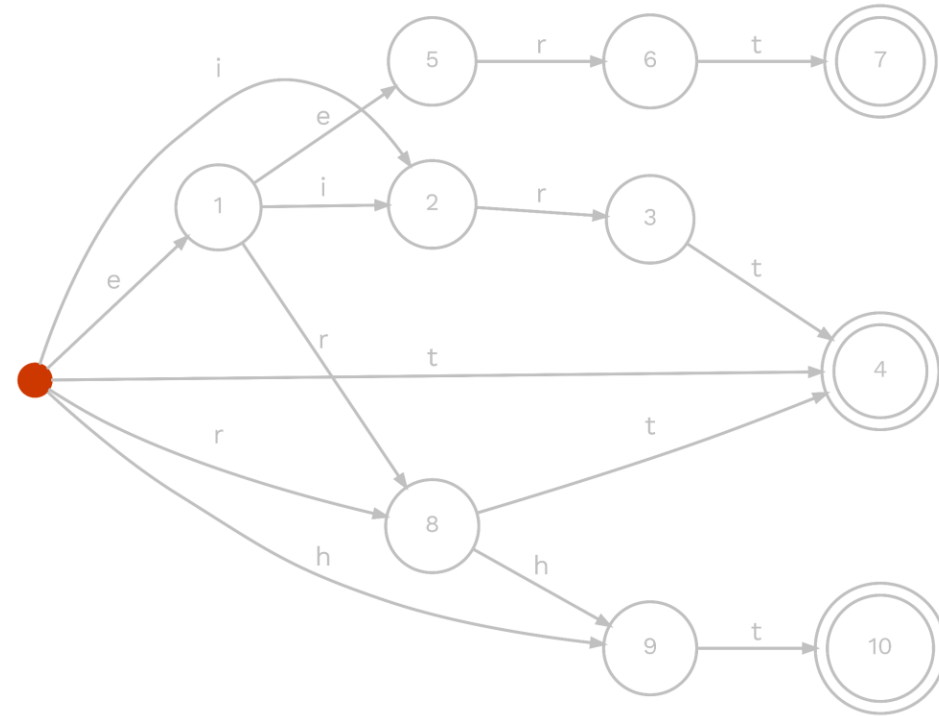
(d) We successfully input *e*, *e*, and *r* in the oracle, but the oracle fail on the character *h*. We shift the window after the *h* and continue our search.



Set Backward Oracle Matching

two_or_three_trees

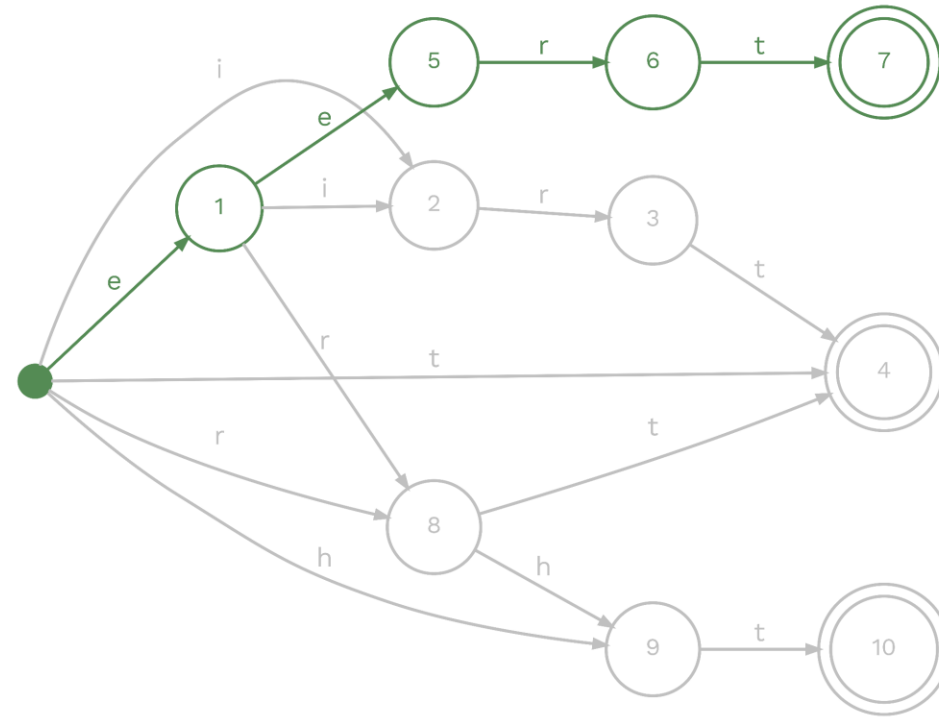
(e) We attempted to input `_` in the factor oracle, but the oracle rejects directly. We shift the window after the `_` and continue our search.



Set Backward Oracle Matching

two_or_three_trees

(f) We input *e*, *e*, *r*, and *t* in the oracle. We reach accepting state 7, which is associated with the word *tree*. Our window contains *tree*.



Applications of FSMs (2)

- **Anomaly detection on system call sequences:**

- Could we use factor oracles for this?

- **Problems:**

- Explosion in size
- Narrowly-defined vs. broad behavior

```

1. S0;
2. while (..) {
3.   S1;
4.   if (...) S2;
5.   else S3;
6.   if (S4) ... ;
7.   else S2;
8.   S5;
9. }
10. S3;
11. S4;

```

$S_0 S_1 S_2$	$S_1 S_2 S_4$	$S_2 S_4 S_5$	$S_3 S_4 S_5$	$S_4 S_5 S_1$	$S_2 S_5 S_1$	$S_5 S_1 S_2$
$S_0 S_1 S_3$	$S_1 S_3 S_4$	$S_2 S_4 S_2$	$S_3 S_4 S_2$	$S_4 S_5 S_3$	$S_2 S_5 S_3$	$S_5 S_1 S_3$
$S_0 S_3 S_4$				$S_4 S_2 S_5$		$S_5 S_3 S_4$

Figure 1. An example program and associated trigrams. S_0, \dots, S_5 denote system calls.

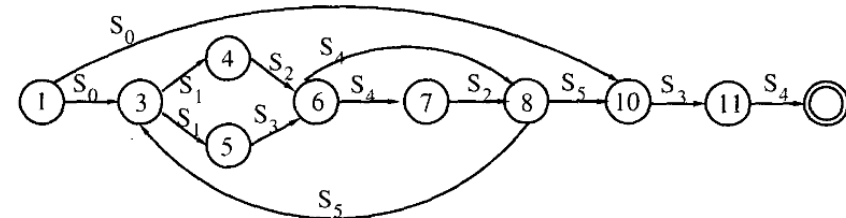


Figure 2. Automaton learnt by our algorithm for Example 1

Outline of the lecture

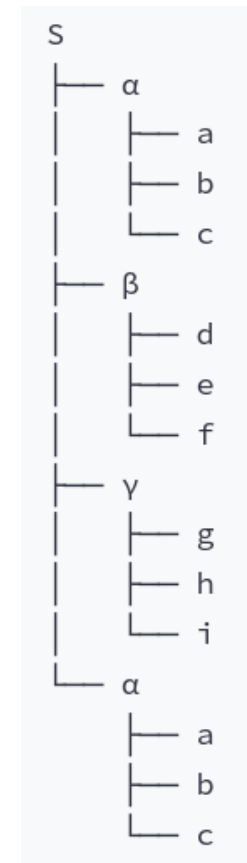
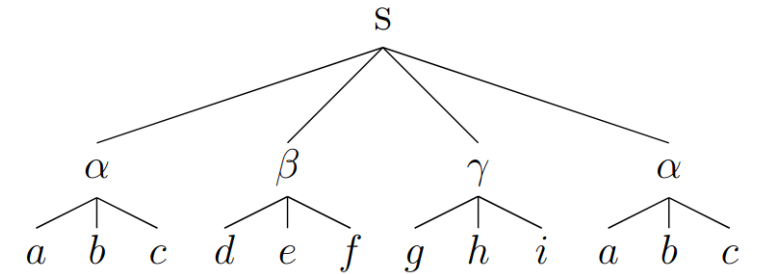
- Preliminary knowledge:
 - Terminology + basics of automata theory
 - Tries, factor automata, and factor oracles
- **Our research at the AI lab:**
 - Hierarchical-alphabet automata (HAAs)
 - Hierarchical factor oracles (HFOs)
- Applications of our research:
 - Anomaly detection with the HFO

Research Goals

- **Solving previously-mentioned issues:**
 - Capturing broad, complex behavior with compact FSMs
- **Possible solution:**
 - Exploiting hierarchical relationships of symbols
- **How?**
 - Words to **hierarchical words** using hierarchical relationships
 - Factors to **hierarchical factors**
 - Finite state machines to **alphabet-hierarchical automata**

Hierarchical Words

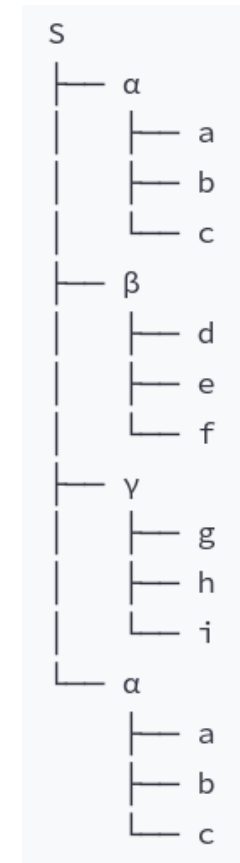
- **Alphabet:** finite set of symbols
- **Hierarchical words:** concatenations of symbols in the form of a rooted tree
- **Hierarchical factors:** factors of hierarchically connected words for every level of the hierarchy



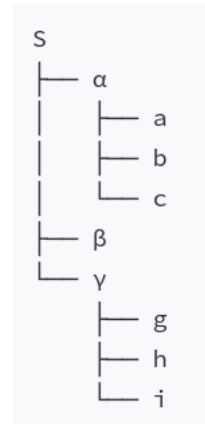
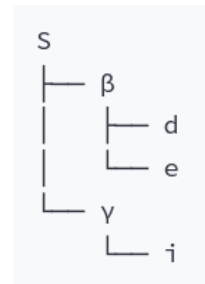
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Hierarchical word:

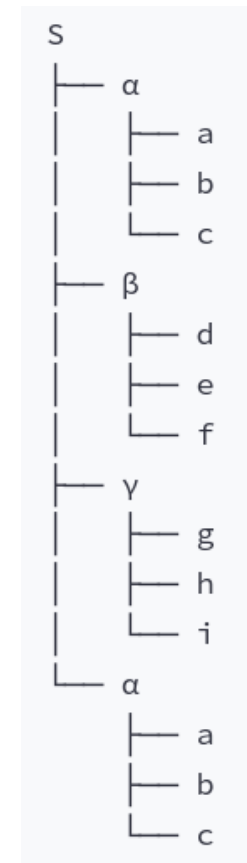
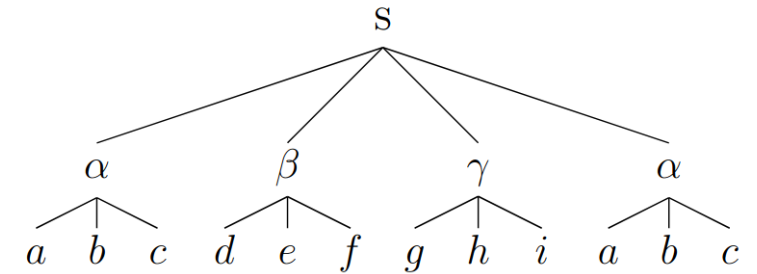


Hierarchical factors:



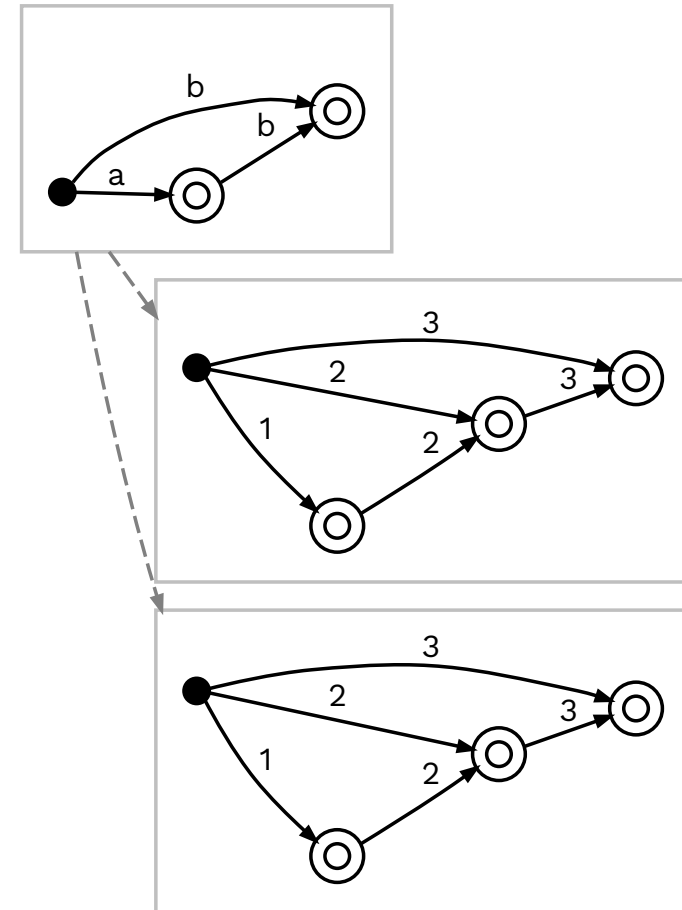
Hierarchical Words

- **Where is hierarchy relevant?**
 - **Language:** words, sentences, paragraphs
 - **Music:** individual notes, chords
 - **Action:** actions into sequence of sub-actions
- **Hierarchy in cybersecurity?**
 - **Parent-child relationships between processes:** one process can spawn others



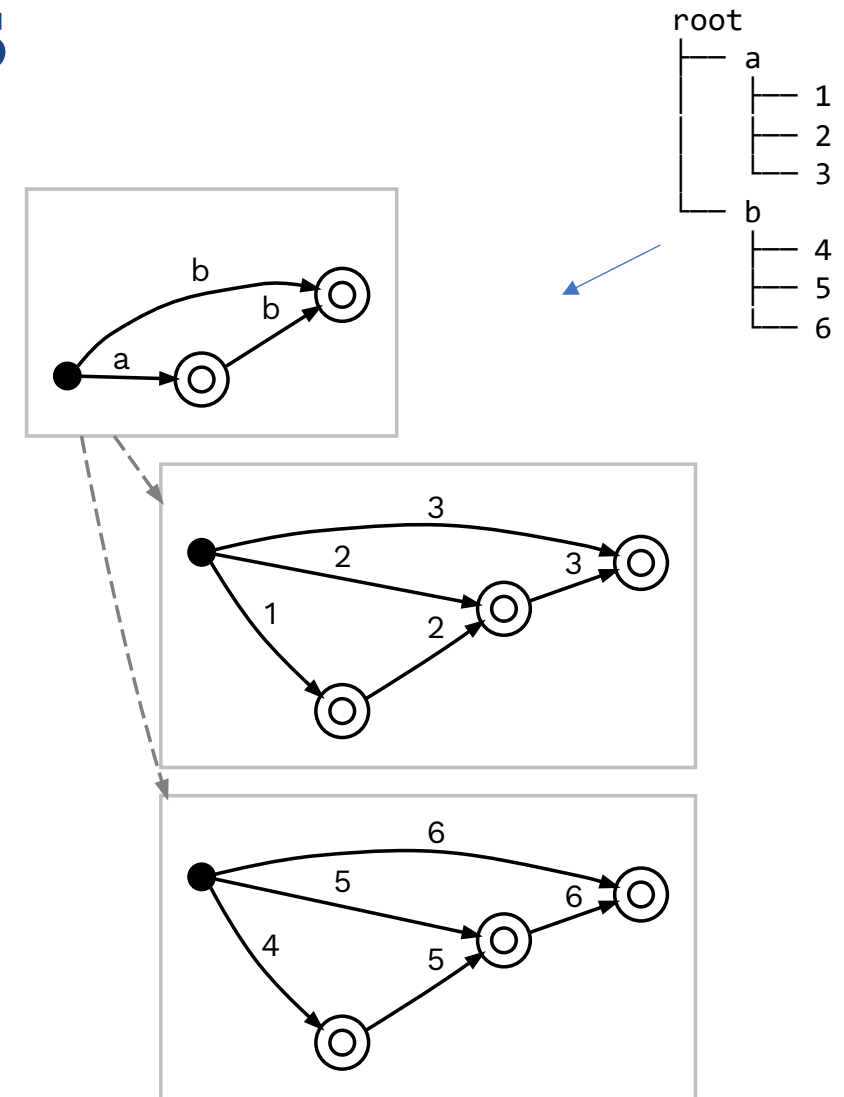
Hierarchical-Alphabet Automata

- **Intuitively:**
 - Hierarchical variant of regular FSMs
 - Has a **super-transition function** Δ that maps symbols of its alphabet in one level to (at most) one other HAA



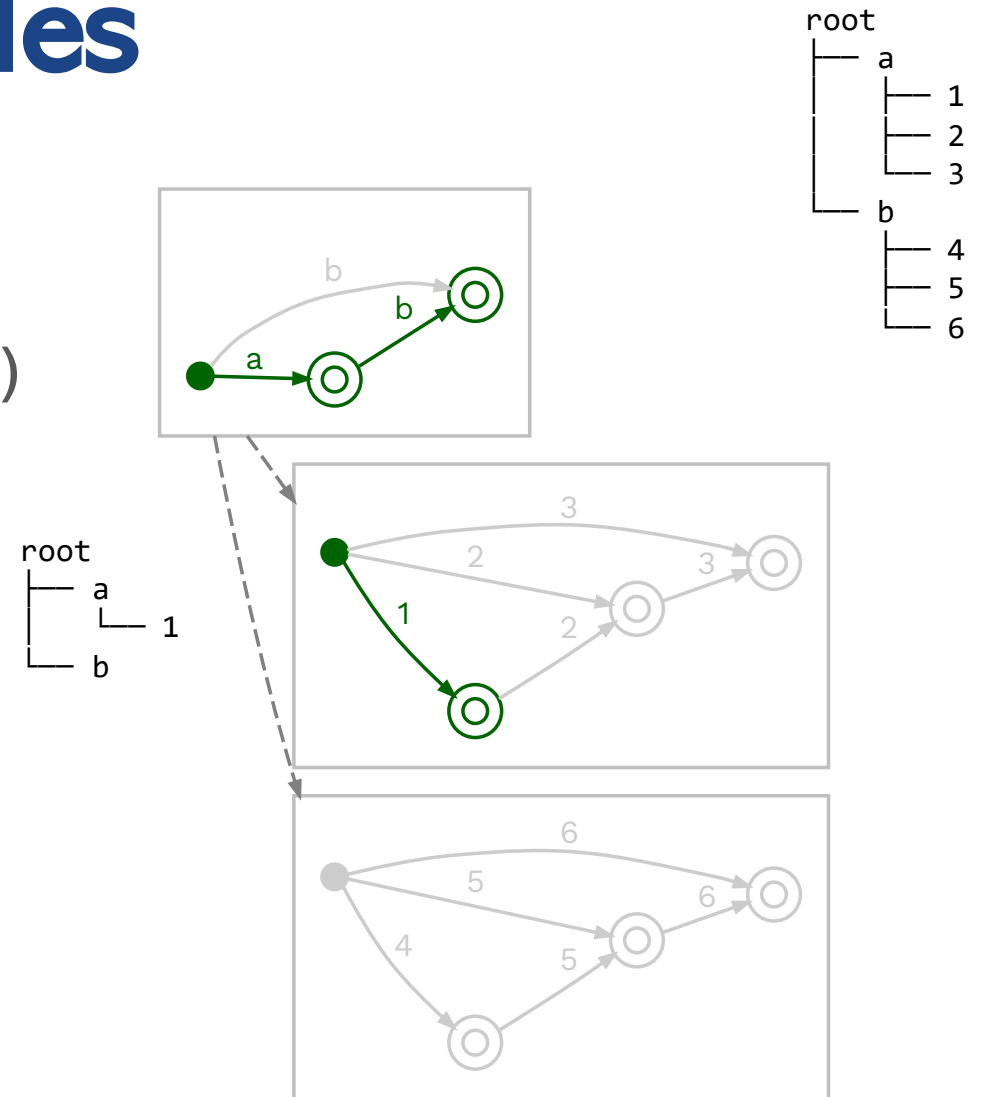
Hierarchical Factor Oracles

- **Hierarchical-alphabet variant of regular (flat) factor oracles:**
 - Built using a (set of) hierarchical word(s)
 - Accepts hierarchical factors of its set



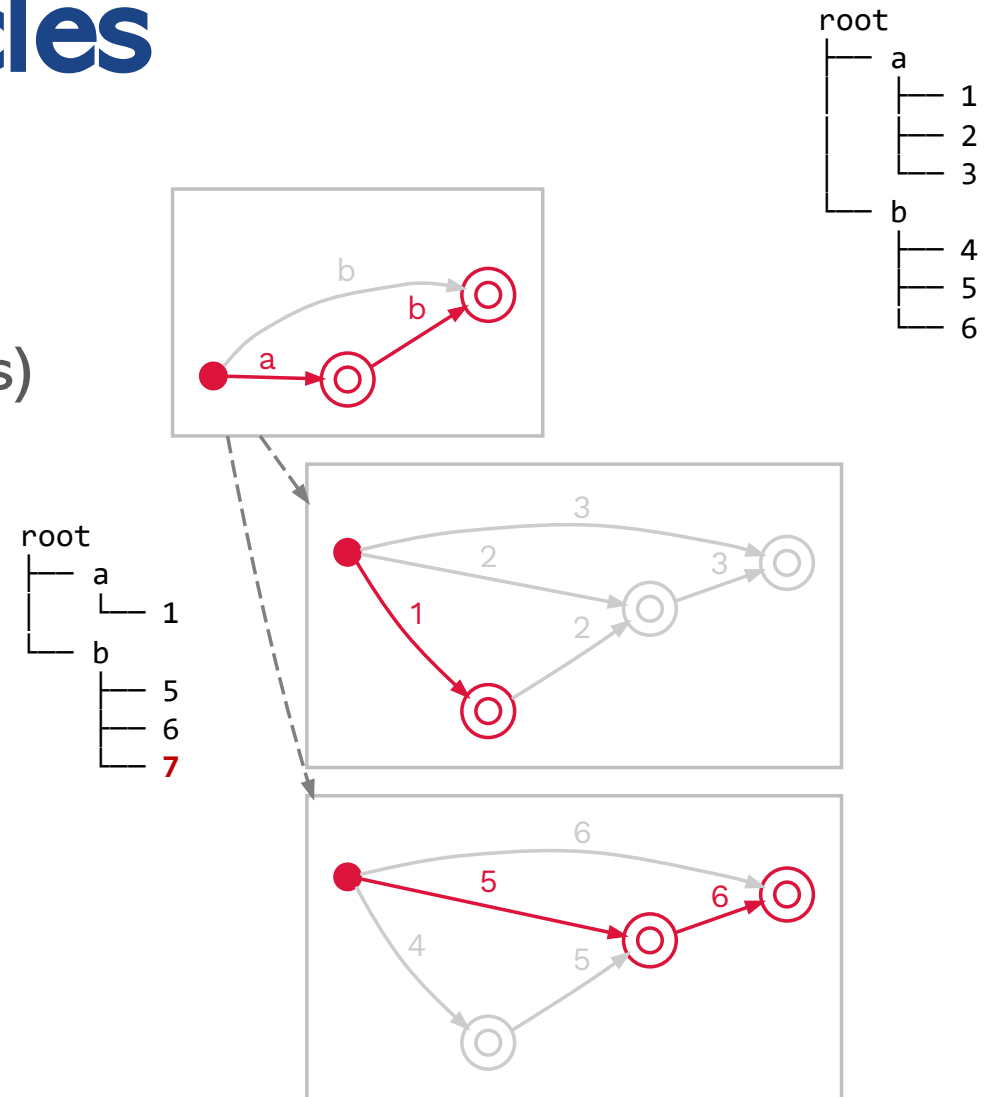
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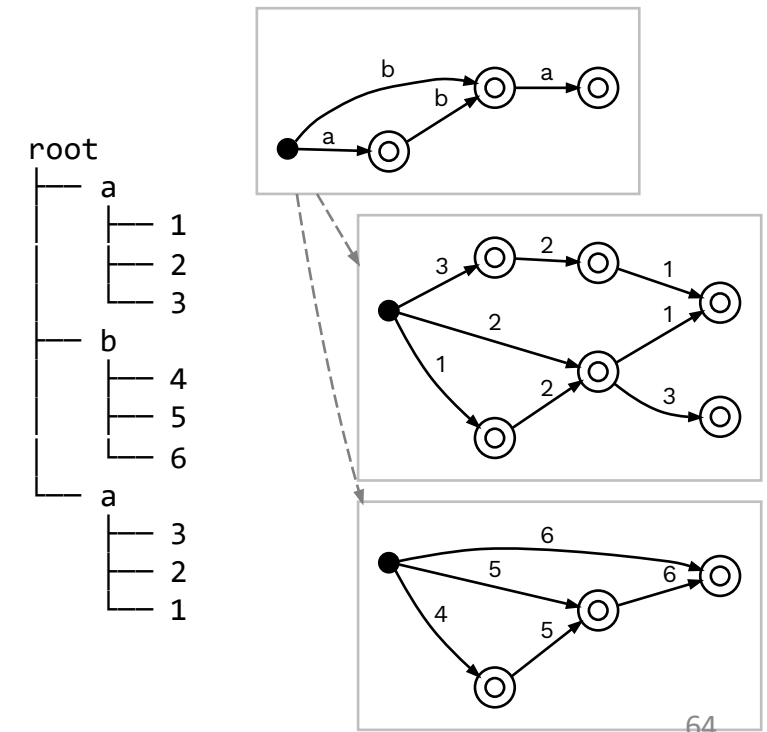
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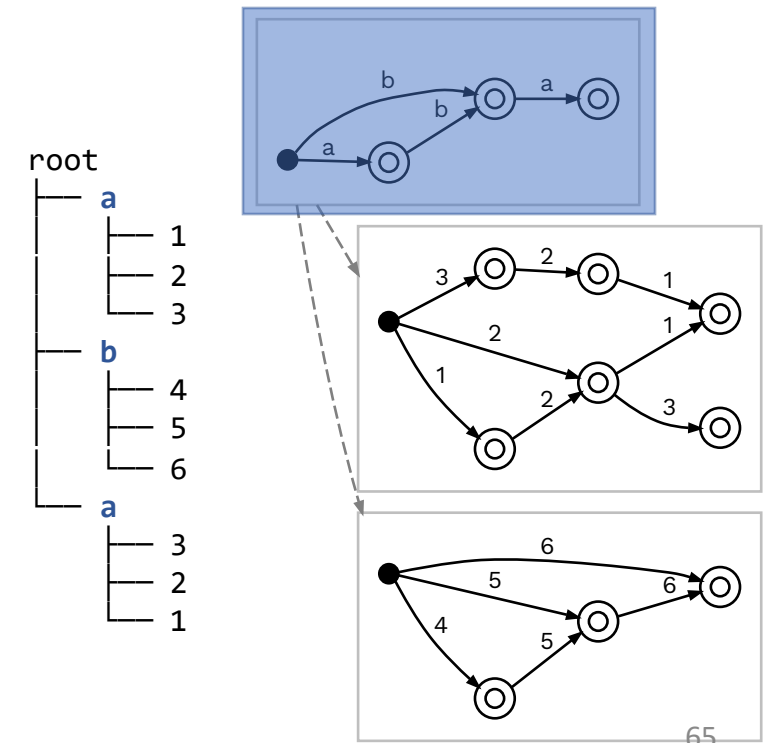
Construction Algorithm for HFOs

- **Hierarchical oracle for a set of hierarchical sequences:**
 - Perform a **breadth-first traversal** on each hierarchical sequence
 - During the traversal, build a **list of regular sequences**
 - **After traversal:** build regular FOs using regular sequences
 - Create **super-transitions from oracle to oracle**



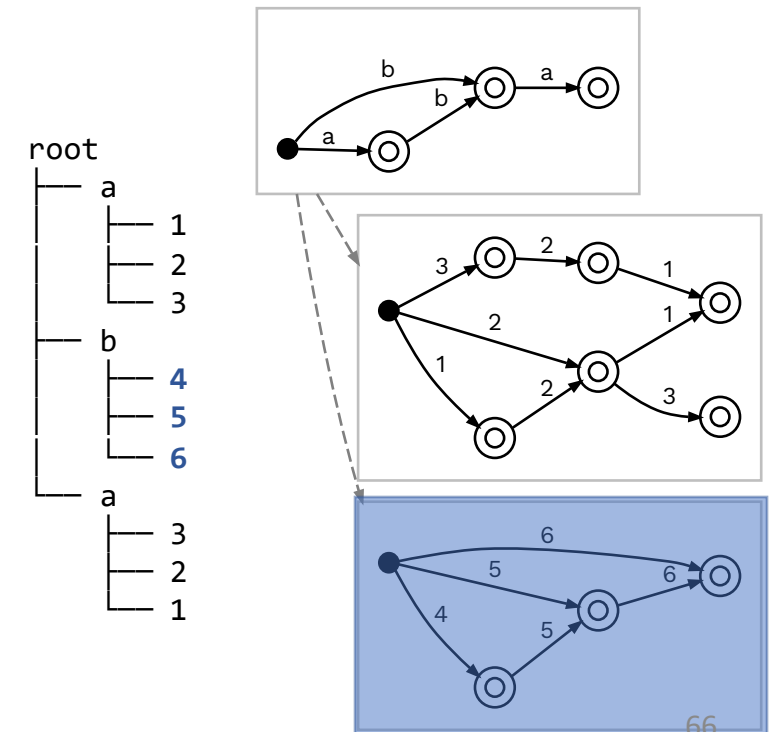
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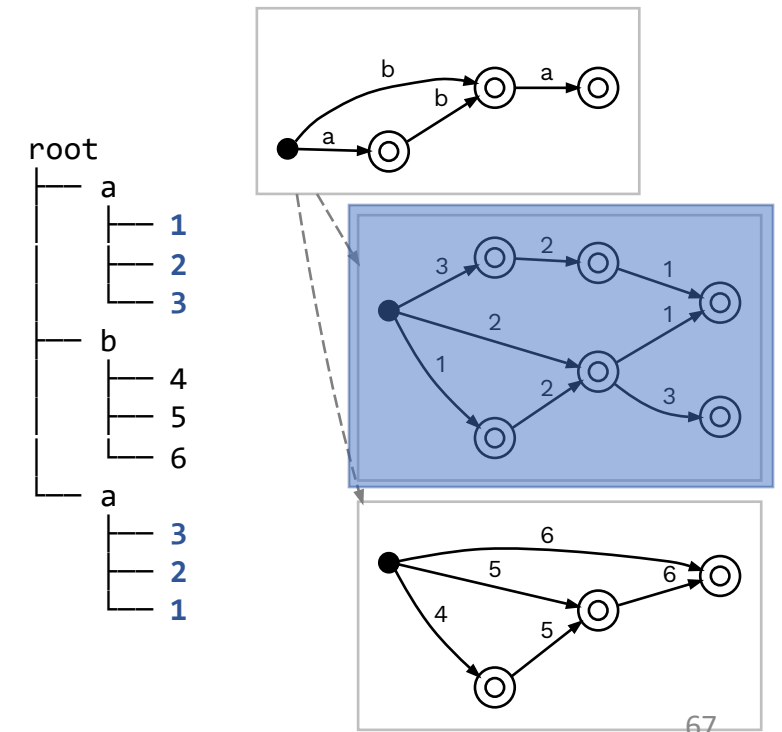
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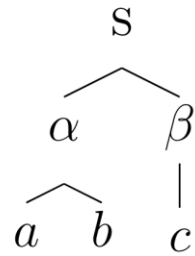
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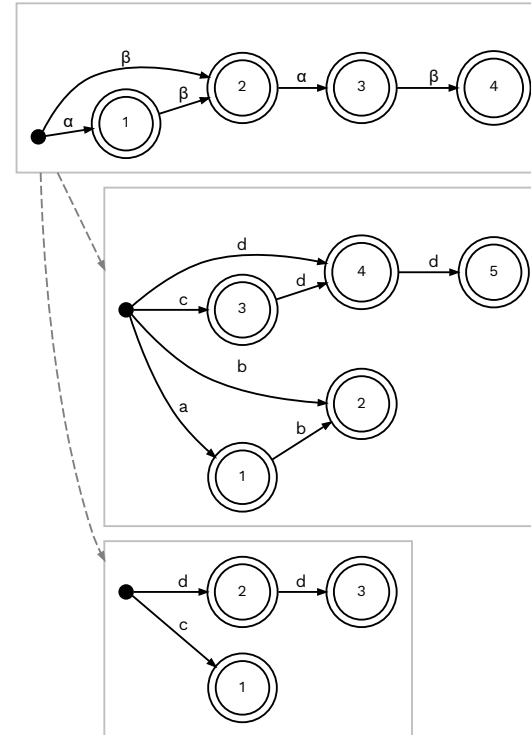


Accepting or rejecting input

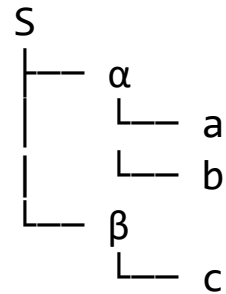
Accepting



with

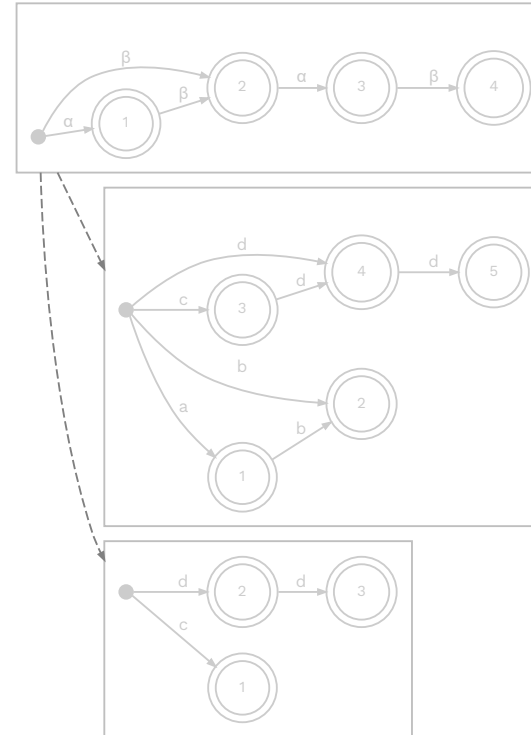


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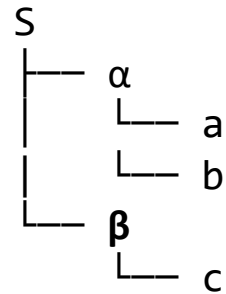


Stack = []

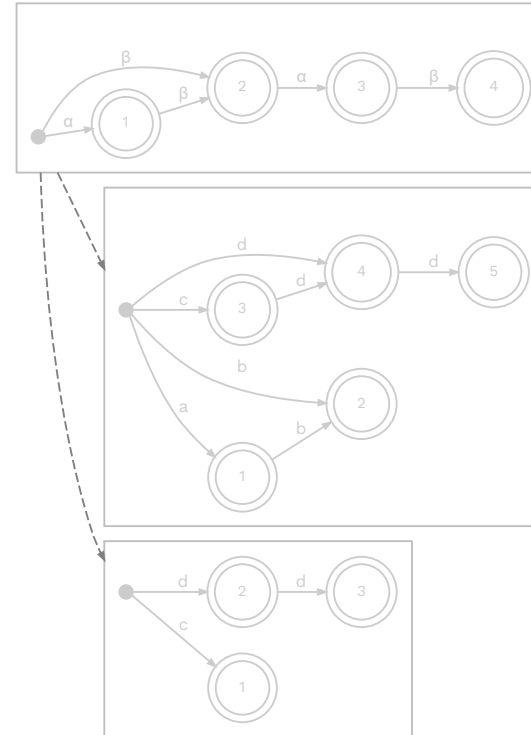
Current = None



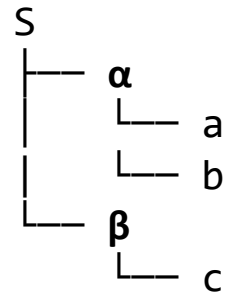
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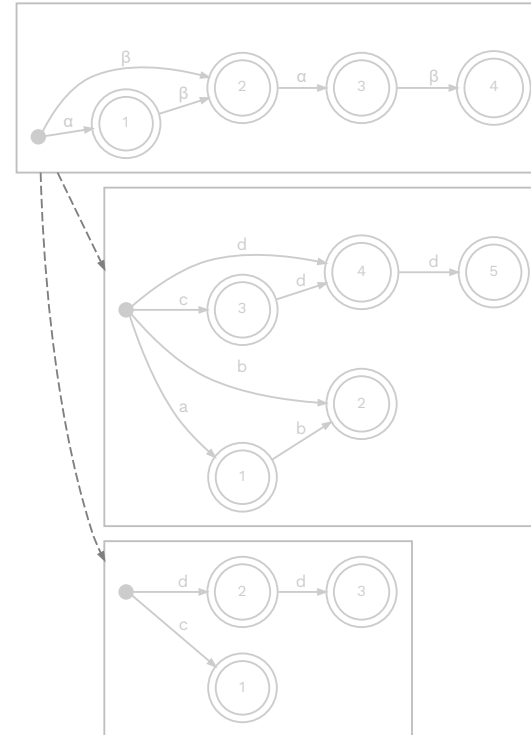
Stack = [β]
Current = None



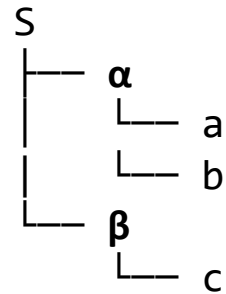
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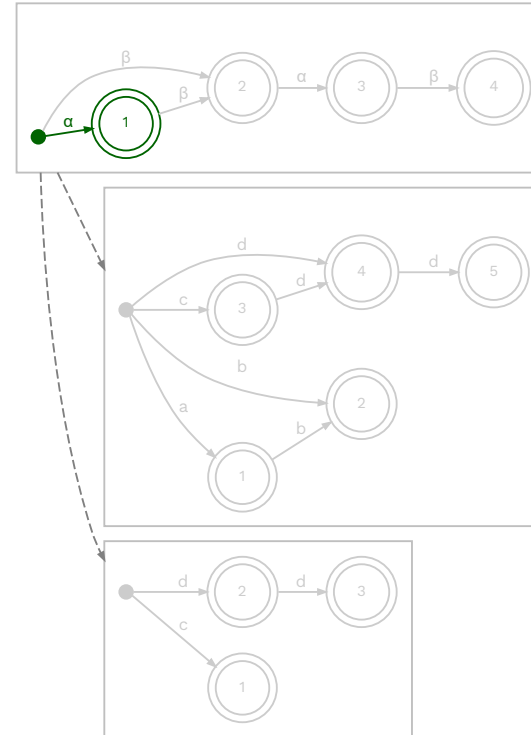
Stack = $[\alpha, \beta]$
Current = None



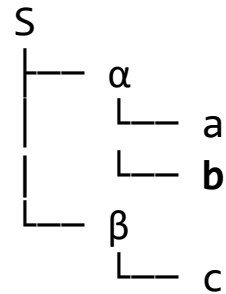
Accepting or rejecting input



Stack = [β]
Current = α

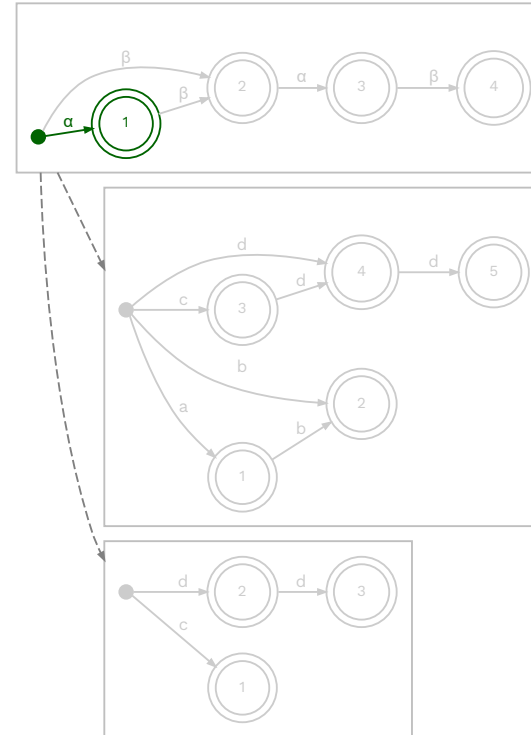


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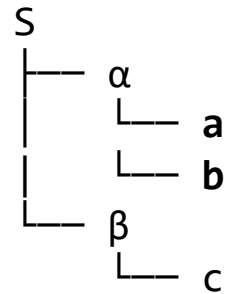


Stack = [b, β]

Current = α

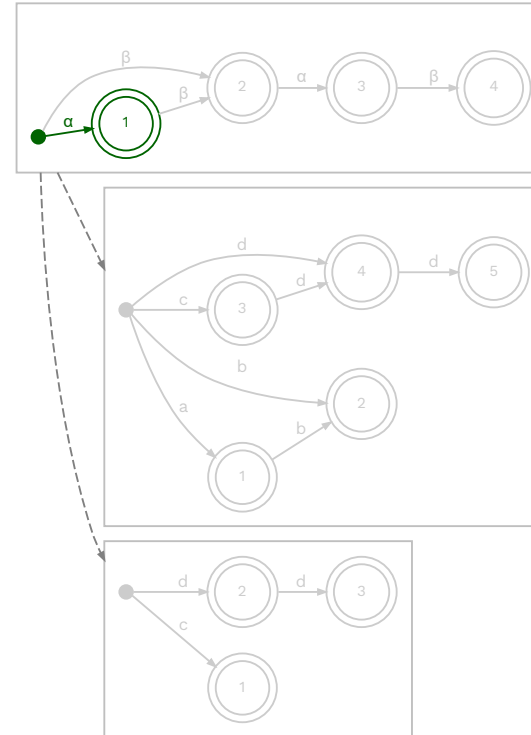


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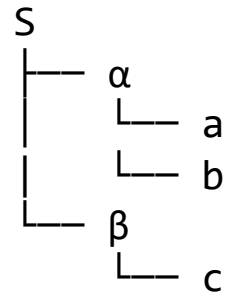


Stack = [a, b, β]

Current = α

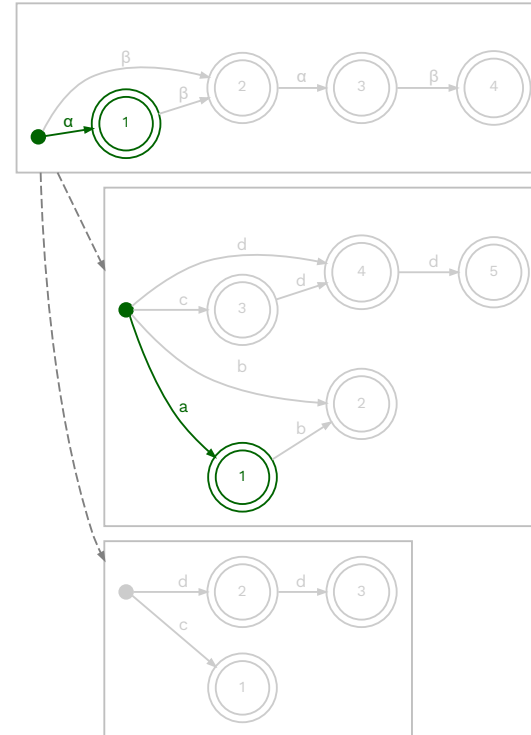


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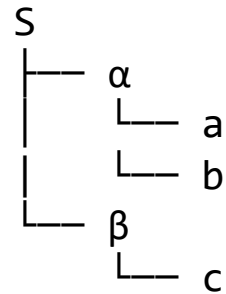


Stack = [b, β]

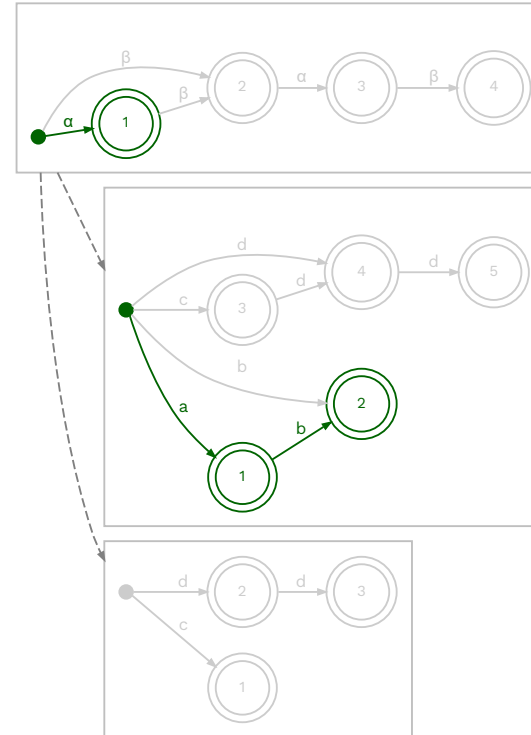
Current = a



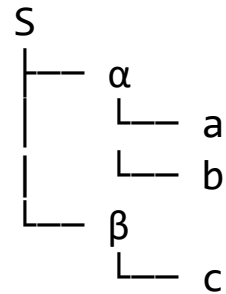
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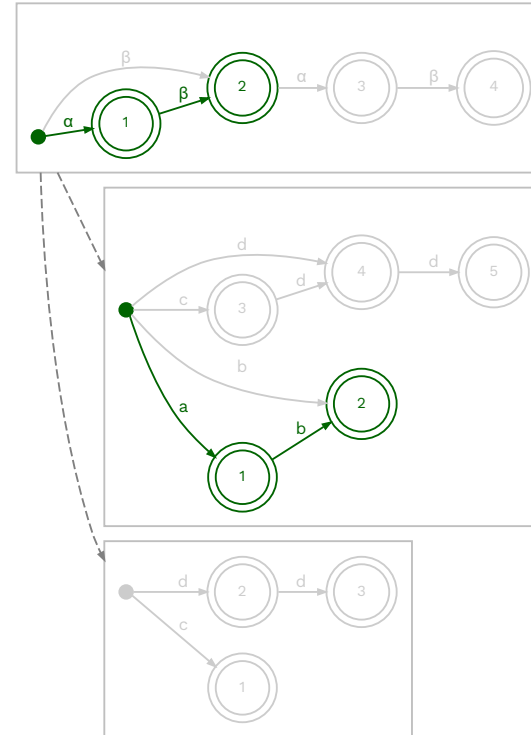
Stack = [β]
Current = b



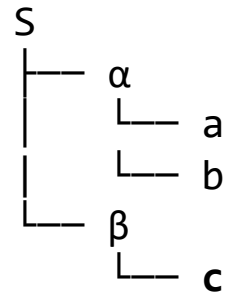
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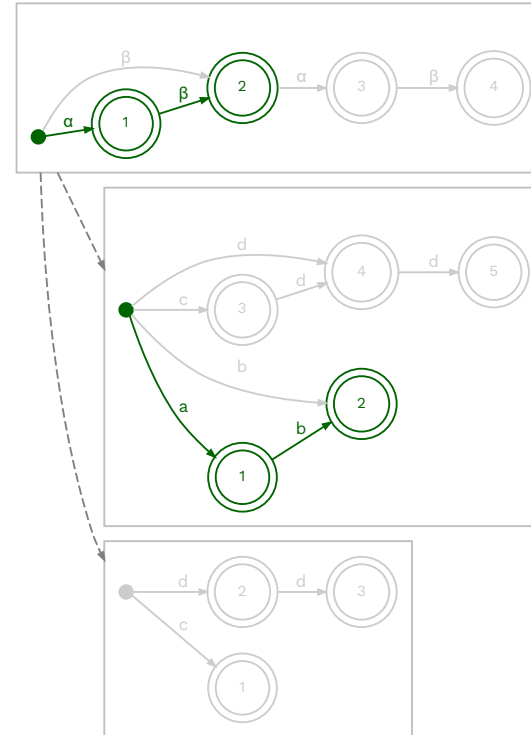
Stack = []
Current = β



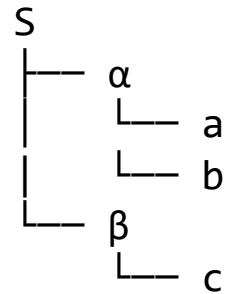
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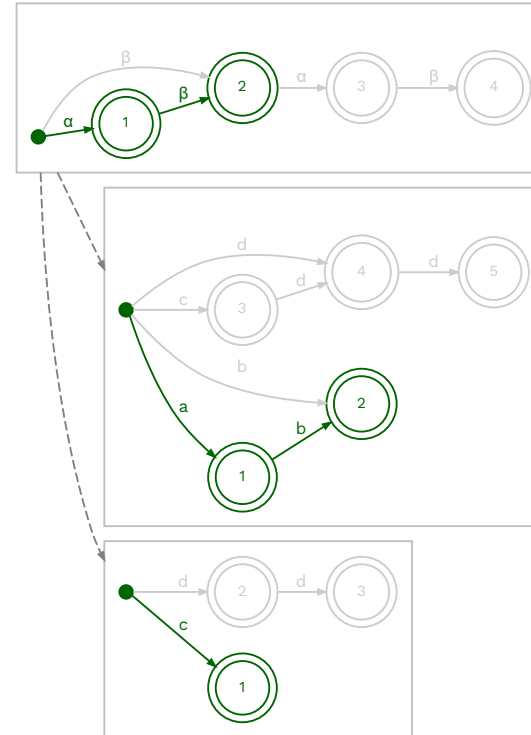
Stack = [c]
Current = β



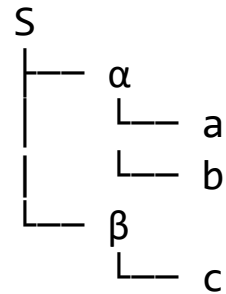
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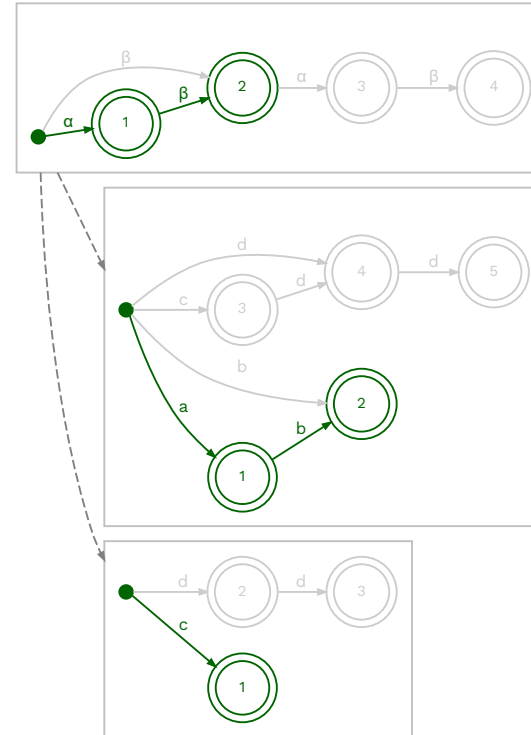


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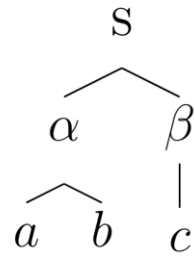


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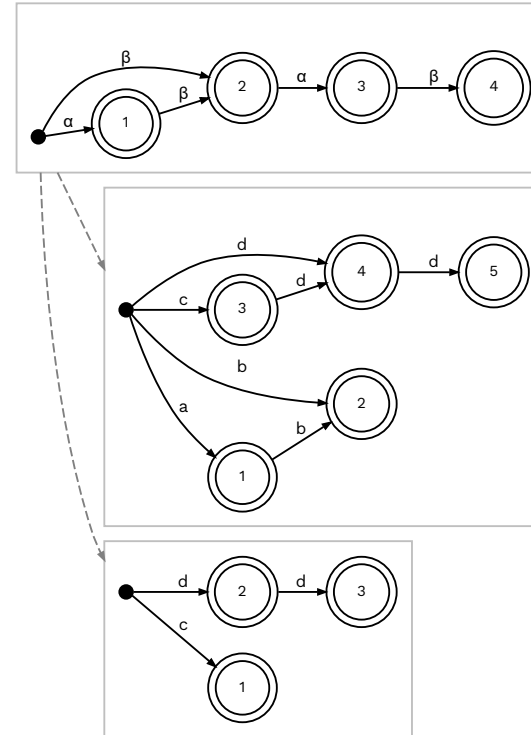
Current = None



Accepting or rejecting input



accepted by



Interactive Demonstration of HFOs

Outline of the lecture

- **Preliminary knowledge:**
 - Terminology + basics of automata theory
 - Tries, factor automata, and factor oracles
- **Our research at the AI lab:**
 - Hierarchical-alphabet automata (HAAs)
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- **Applications of our research:**
 - Anomaly detection with the HFO

Anomaly Detection

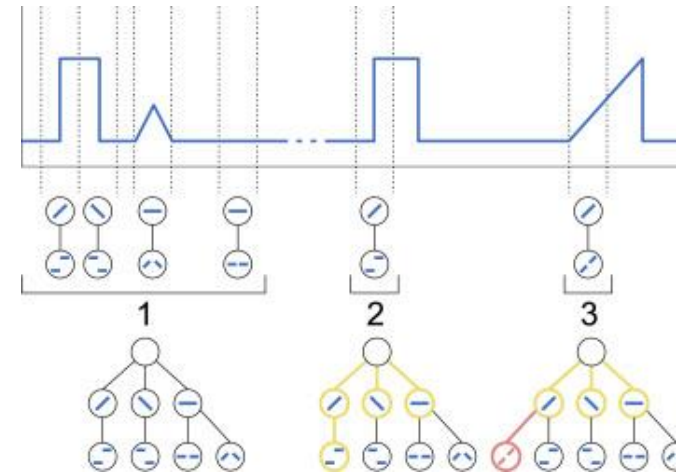
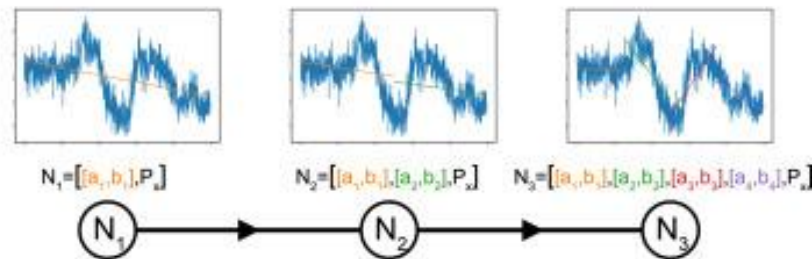
- **What is it?**
 - Finding patterns in data that do not conform to expected behavior
 - Relevant in lots of domains and applications, and real-life relevance!
- **(Time) series anomaly detection:**
 - Detecting spikes, drops, out-of-place patterns, unexpected trends, ...

HPM-based Anomaly Detection

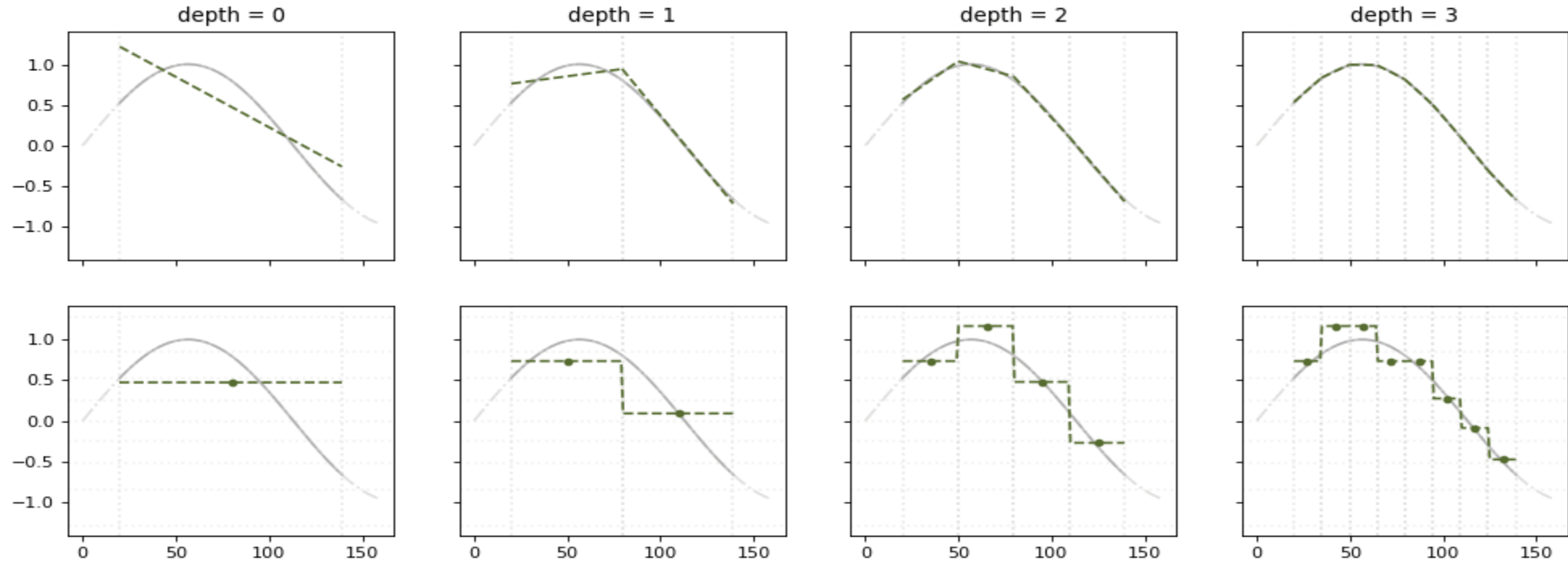
- **Main focus: anomaly detection in production environments**
 - Needs to be **lightweight** due to constraints in resources
 - We should never forget **rarely-occurring events**
 - We should be able to adapt to **new behavior**
 - It should be **general** enough to work on different time series
 - It should **recognize non-strictly periodic events**
 - Should have a **minimum amount of false positives**

HPM-based Anomaly Detection

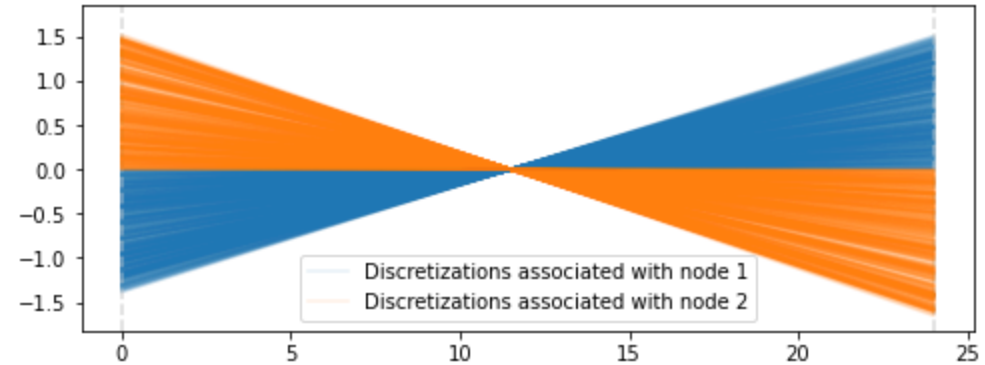
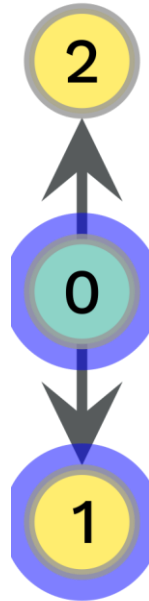
- **Key idea:** extract hierarchical fingerprints from time series
 - Store all seen fingerprints in a tree structure
 - New fingerprints are marked as possible anomalies



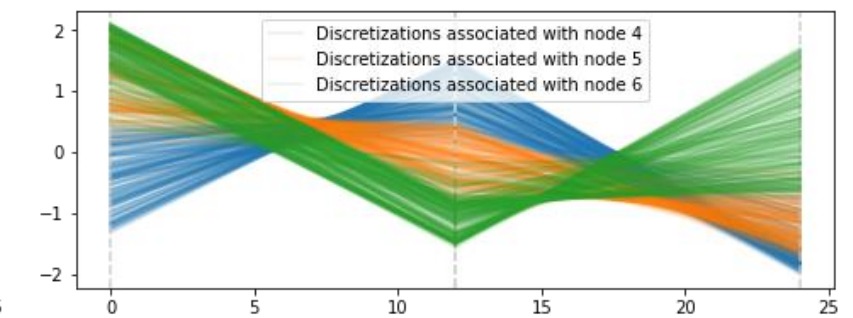
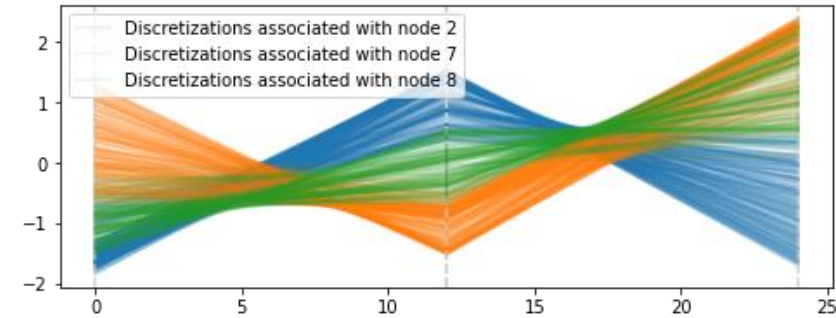
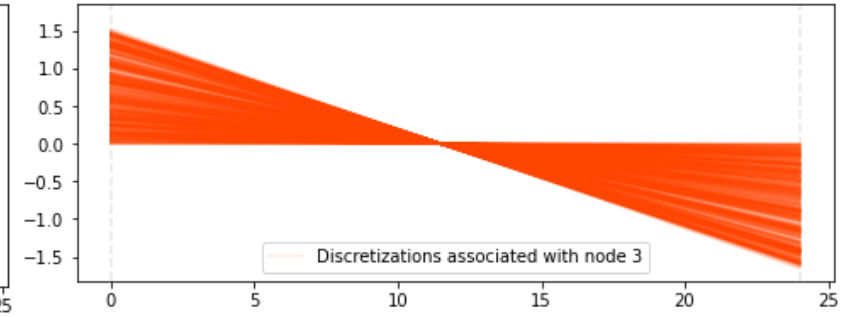
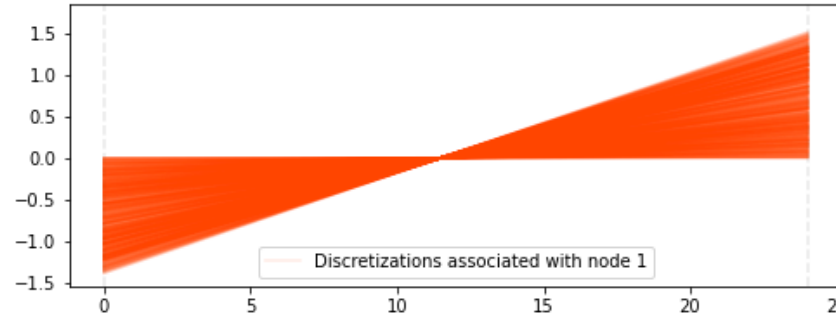
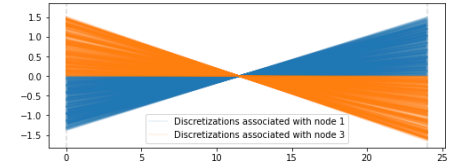
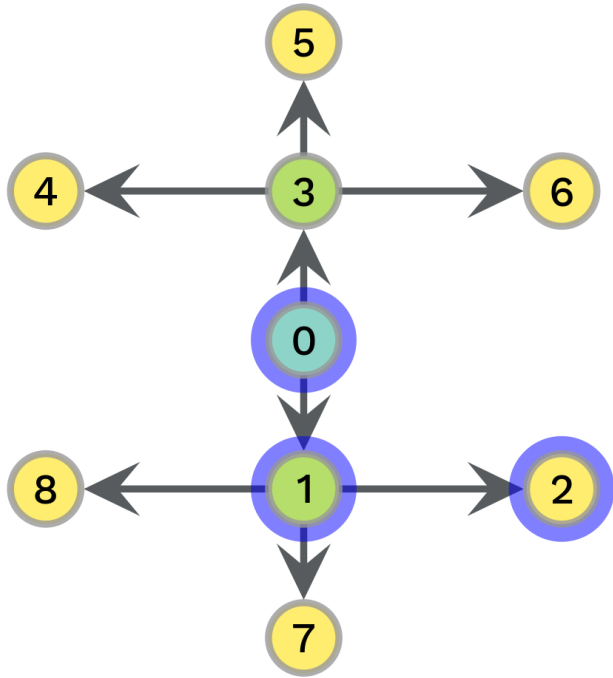
HPM-based Anomaly Detection



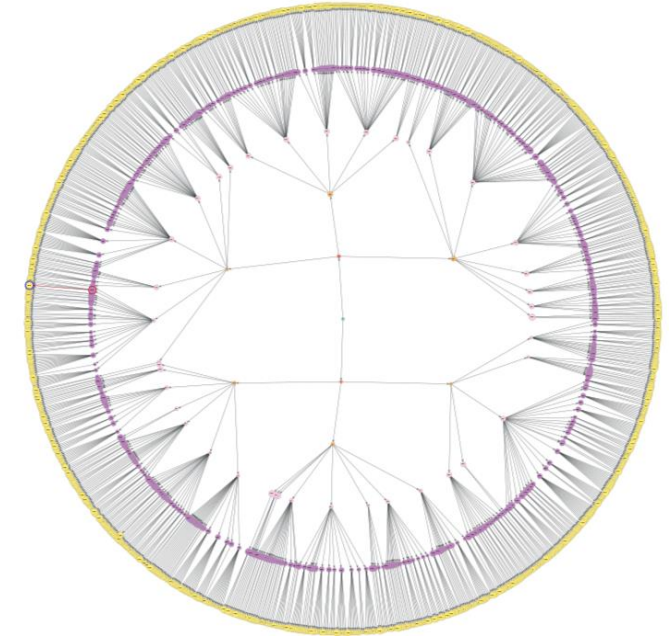
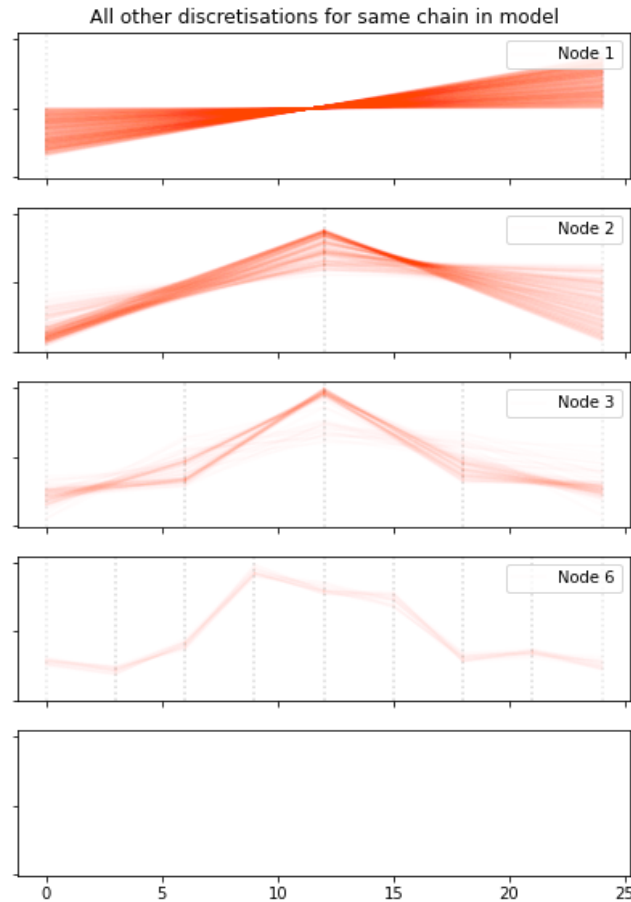
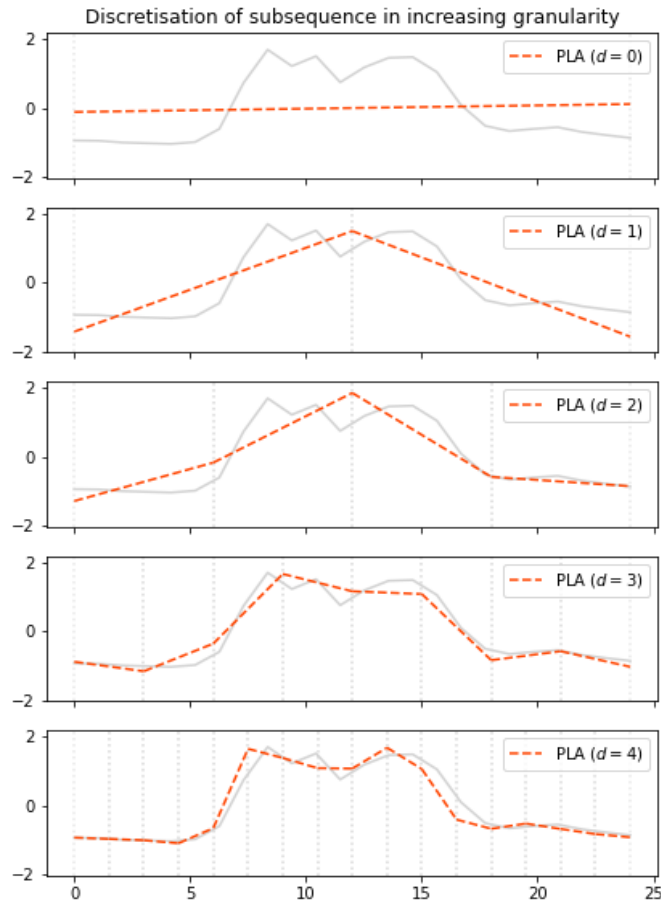
HPM-based Anomaly Detection



HPM-based Anomaly Detection



HPM-based Anomaly Detection



Interactive Demonstration of HPM

HPM-based Anomaly Detection

- **Disadvantages of basic algorithm:**
 - Everything is anomalous if you go fine-grained enough
 - No temporal anomalies: only new patterns are possible anomalies
 - Lack of context: have we seen a factor of fingerprints before?

Extending HPM with HFOs

- **Possible approach:**
 1. **Segmenting:** extract subsequences using a sliding window
 2. **Discretizing:** extract fingerprints from above-obtained subsequences
 3. **Learning:** represent consecutive fingerprints as hierarchical words (and HFO)
 4. **Anomaly detection:** calculate score based on largest accepted HFO input

Extending HPM with HFOs

- **Segmenting:**

		requests						
2022-07-31	22:00:00+00:00	29309						
2022-07-31	23:00:00+00:00	29977	[29309	29977	29748	...	40237 36560 34145]
2022-08-01	00:00:00+00:00	29748	[29977	29748	28414	...	36560 34145 31998]
2022-08-01	01:00:00+00:00	28414	[29748	28414	27871	...	34145 31998 31041]
2022-08-01	02:00:00+00:00	27871						
...								
2022-10-14	17:00:00+00:00	41931	[37564	34118	29764	...	41931 41215 38529]
2022-10-14	18:00:00+00:00	41215	[34118	29764	29600	...	41215 38529 37104]
2022-10-14	19:00:00+00:00	38529	[29764	29600	29485	...	38529 37104 34412]
2022-10-14	20:00:00+00:00	37104						
2022-10-14	21:00:00+00:00	34412						

Extending HPM with HFOs

- **Discretizing:**

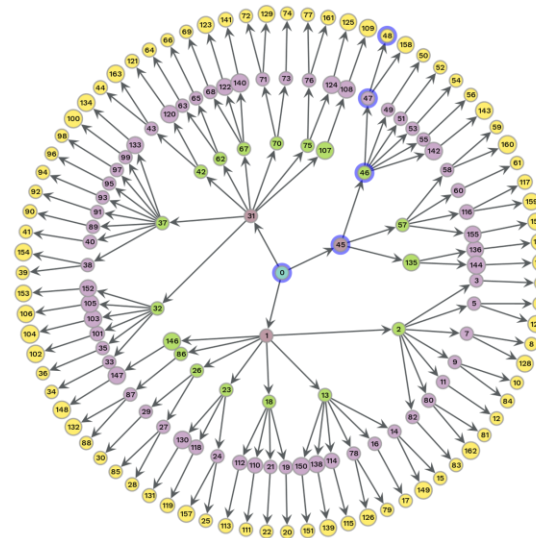
```
[[29309 29977 29748 ... 40237 36560 34145]  
 [29977 29748 28414 ... 36560 34145 31998]  
 [29748 28414 27871 ... 34145 31998 31041]  
 ...  
 [37564 34118 29764 ... 41931 41215 38529]  
 [34118 29764 29600 ... 41215 38529 37104]  
 [29764 29600 29485 ... 38529 37104 34412]]
```

```
[[Node(depth, slope, inter, ...), Node(...), ...]  
 [Node(depth, slope, inter, ...), Node(...), ...]  
 [Node(depth, slope, inter, ...), Node(...), ...]  
 ...  
 [Node(depth, slope, inter, ...), Node(...), ...]  
 [Node(depth, slope, inter, ...), Node(...), ...]  
 [Node(depth, slope, inter, ...), Node(...), ...]]
```

Extending HPM with HFOs

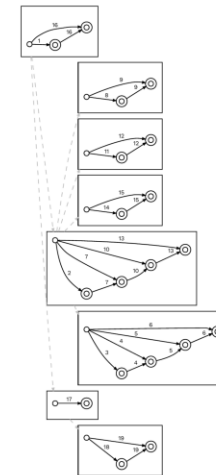
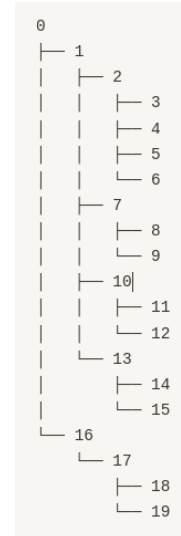
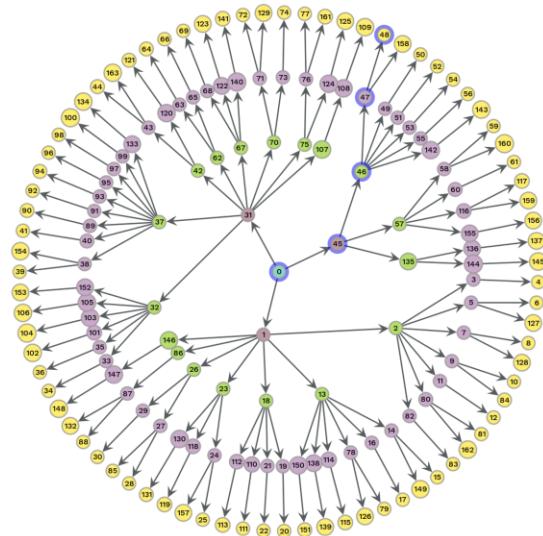
- **Learning:**

```
[[Node(depth, slope, inter), Node(...), ...]  
 [Node(depth, slope, inter), Node(...), ...]  
 [Node(depth, slope, inter), Node(...), ...]  
 ...  
 [Node(depth, slope, inter), Node(...), ...]  
 [Node(depth, slope, inter), Node(...), ...]  
 [Node(depth, slope, inter), Node(...), ...]]
```



Extending HPM with HFOs

- Learning:



Extending HPM with HFOs

- **Advantages over HPM:**
 - Not everything is anomalous: search finds largest accepting factor
 - Temporal anomalies: old patterns are possibly anomalous
 - Lack of context: checks for *factors* of fingerprints
- **Advantages over HPM (extended with other state methods):**
 - Compactness: exploiting hierarchy of discretization for compactness
 - Hierarchy: takes hierarchy of discretization into account

Interactive Demonstration of Algorithm

