Computational Learning Theory

[read Chapter 7]
[Suggested exercises: 7.1, 7.2, 7.5, 7.8]

- Computational learning theory
- Setting 1: learner poses queries to teacher
- Setting 2: teacher chooses examples
- Setting 3: randomly generated instances, labeled by teacher
- Probably approximately correct (PAC) learning
- Vapnik-Chervonenkis Dimension
- Sample Complexity
- Computational Complexity
- Mistake bounds

Computational Learning Theory

What general laws constrain inductive learning?

We seek theory to relate:

- Probability of successful learning, δ
- Number of training examples, m
- Complexity of hypothesis space, $|H| or |VC|$
- Accuracy to which target concept is approximated, ϵ
- Manner in which training examples presented at random

Prototypical Concept Learning Task

- **Given:**
  - Instances X: Possible days, each described by the attributes *Sky, AirTemp, Humidity, Wind, Water, Forecast*
  - Target function c: $EnjoySport : X \rightarrow \{0,1\}$
  - Hypotheses H: Conjunctions of literals. E.g.

    $$\langle ?, Cold, High, ?, ?, ? \rangle.$$

  - Training examples D: Positive and negative examples of the target function

    $$\langle x_1, c(x_1) \rangle, \ldots \langle x_m, c(x_m) \rangle$$

- **Determine:**
  - A hypothesis h in H such that $h(x) = c(x)$ for all x in trainingdata D?
  - A hypothesis h in H such that $h(x) = c(x)$ for all x in X?

Sample Complexity

How many training examples are sufficient to learn the target concept?

1. If learner proposes instances, as queries to teacher
   - Learner proposes instance x, teacher provides $c(x)$
2. If teacher (who knows c) provides training examples
   - teacher provides sequence of examples of form $\langle x, c(x) \rangle$
3. If some random process (e.g., nature) proposes instances
   - instance x generated randomly, teacher provides $c(x)$

Sample Complexity: 1

Learner proposes instance x, teacher provides $c(x)$

(assume c is in learner's hypothesis space H)

Optimal query strategy: play 20 questions

- pick instance x such that half of hypotheses in VS classify x positive, half classify x negative
- When this is possible, need $\lceil \log_2 |H| \rceil$ queries to learn c
- when not possible, need even more

Sample Complexity: 2

Teacher (who knows c) provides training examples

(assume c is in learner's hypothesis space H)

Optimal teaching strategy: depends on H used by learner

Consider the case $H =$ conjunctions of up to n boolean literals and their negations

e.g., $(AirTemp = Warm) \wedge (Wind = Strong)$, where $AirTemp, Wind, \ldots$ each have 2 possible values.

- if n possible boolean attributes in H, $n + 1$ examples suffice
- why?

Sample Complexity: 3

Given:

- set of instances X
- set of hypotheses H
- set of possible target concepts C
- training instances generated by a fixed, unknown probability distribution $\mathcal{D}$ over X

Learner observes a sequence D of training examples of form $\langle x, c(x) \rangle$, for some target concept $c \in C$

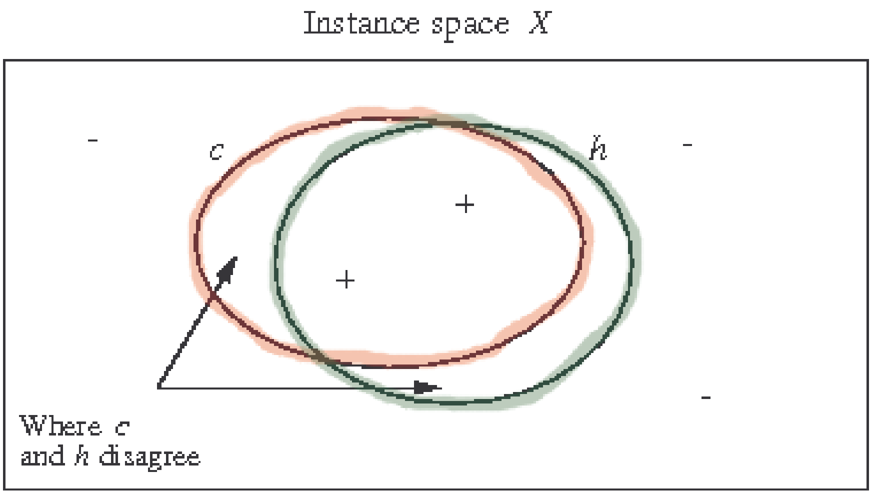
- instances x are drawn from distribution $\mathcal{D}$
- teacher provides target value $c(x)$ for each

Learner must output a hypothesis h estimating c

- h is evaluated by its performance on subsequent instances drawn according to $\mathcal{D}$

Note: randomly drawn instances, noise-free classifications

True Error of a Hypothesis



Definition: The **true error** (denoted $error_{\mathcal{D}}(h)$) of hypothesis h with respect to target concept c and distribution $\mathcal{D}$ is the probability that h will misclassify an instance drawn at random according to $\mathcal{D}$.

$$error_{\mathcal{D}}(h) \equiv \Pr_{x \in \mathcal{D}}[c(x) \neq h(x)]$$

Two Notions of Error

Training error of hypothesis h with respect to target concept c

- How often $h(x) \neq c(x)$ over training instances

True error of hypothesis h with respect to c

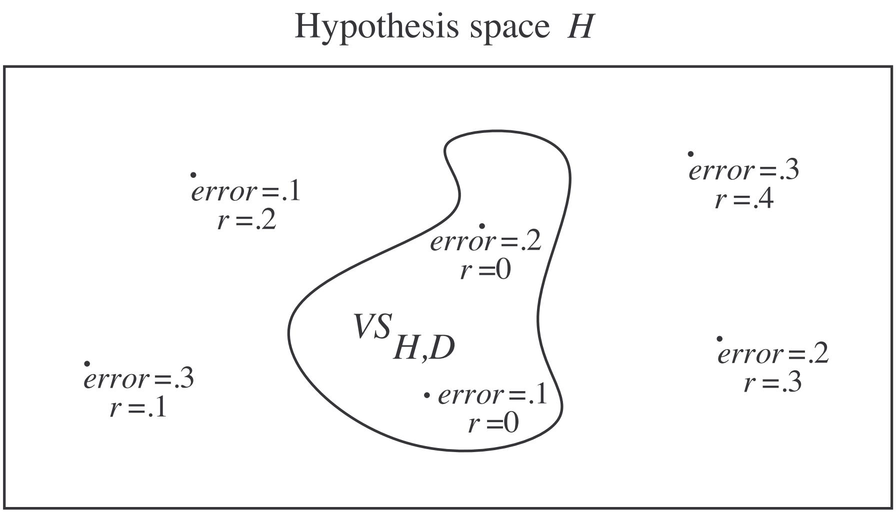
- How often $h(x) \neq c(x)$ over future random instances

Our concern:

- Can we bound the true error of h given the training error of h?
- First consider when training error of h is zero (i.e., $h \in VS_{H,D}$)

$\rightarrow$ Consistent Learners

Exhausting the Version Space



(r = training error, $error$ = true error)

<u>**Definition:**</u> The version space $VS_{H,D}$ is said to be ϵ-**exhausted** with respect to c and $\mathcal{D}$, if every hypothesis h in $VS_{H,D}$ has error less than ϵ with respect to c and $\mathcal{D}$.

$$(\forall h \in VS_{H,D})\ error_{\mathcal{D}}(h) < \epsilon$$

How many examples will ϵ-exhaust the VS?

Theorem: [Haussler, 1988].

If the hypothesis space H is finite, and D is a sequence of $m \geq 1$ independent random examples of some target concept c, then for any $0 \leq \epsilon \leq 1$, the probability that the version space with respect to H and D is not ϵ-exhausted (with respect to c) is less than

$$|H|e^{-\epsilon m}$$

Interesting! This bounds the probability that any consistent learner will output a hypothesis h with $error(h) \geq \epsilon$

If we want to this probability to be below δ

$$|H|e^{-\epsilon m} \leq \delta$$

then

$$m \geq \frac{1}{\epsilon}(\ln |H| + \ln(1/\delta))$$

Learning Conjunctions of Boolean Literals

How many examples are sufficient to assure with probability at least $(1 - \delta)$ that

every h in $VS_{H,D}$ satisfies $error_{\mathcal{D}}(h) \leq \epsilon$

Use our theorem:

$$m \geq \frac{1}{\epsilon}(\ln |H| + \ln(1/\delta))$$

Suppose H contains conjunctions of constraints on up to n boolean attributes (i.e., n boolean literals). Then $|H| = 3^n + 1$, and

$$m \geq \frac{1}{\epsilon}(\ln(3^n + 1) + \ln(1/\delta))$$

or

$$m \geq \frac{1}{\epsilon}(n \ln 3 + \ln(1/\delta))$$

Linear in n and $\frac{1}{n}$