SHORTEST PATH PROBLEM IN INTERMODAL GRAPHS

SOFTWARE AIDS FOR RESOLVING MOBILITY PROBLEMS IN BRUSSELS

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INTRODUCTION

Aim of my research:

Create **shortest path algorithms** on the city of **Brussels** that are:

1. **Intermodal** (may contain several modalities in a unique solution/path)
2. **User-adapted** (different users have different solutions)
3. Take into account **all the transports** on Brussles (there is more than 20 private and public transport's actors on Brussels)

Tasks:

1. Centralize the **data** (roads map and transports) in a unique intermodal graph
2. Create **algorithms** that can make intermodal searches

RESUME OF THE LECTURE

I. Introduction to shortest path problem

II. Intermodal and user-adapted path search

III. Optimization methods

SHORTEST PATH PROBLEM

Shortest path problem is a well known problem of **Combinatorial optimization.**

From a theoritical point of view, almost all combinatorial optimization's problems can be solved as following:

1. Compute all possibilities
2. Take the optimal one

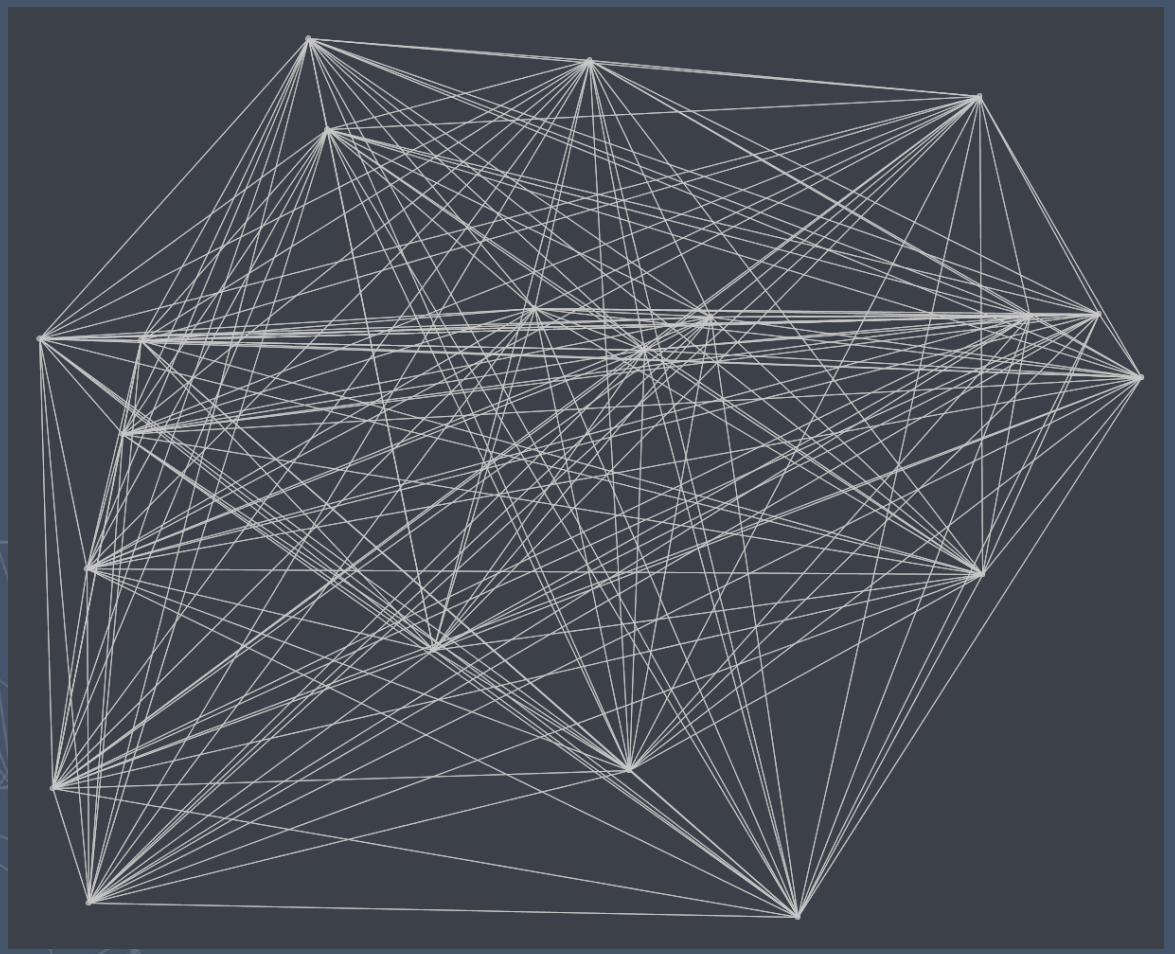
Problem: **Combinatorial explosion**

SHORTEST PATH PROBLEM

Example:

For a **complete graph** with only **20** vertices, there are **17 403 456 103 284 420** possible path between two points.

So, the art of combinatorial optimization is to find **« smart » ways** to compute the minimal amount of combinations and still find the optimal solution, in order to **make the computation possible**.



SHORTEST PATH PROBLEM

Consider a **directed weighted graph** [1] $G = (V, E, l)$ where V is the set of **vertices**, E is the set of **edges** and l is a **weight function** $l\colon\ E \to \mathbb{R}$.

For simplicity we can **note** an edge e as $(\boldsymbol{v}, \boldsymbol{v'})$ where v is its **stat** vertice and v' its **end** vertice.

A **path** is a **sequence** $\mathrm{P} = \{e_1, e_2, \dots, e_k\}$ of **edges** such that the end vertice of e_i is the start vertice of e_{i+1} for $i = 1, \dots, k-1$.

For two vertices s and t, a $\{\boldsymbol{s}, \boldsymbol{t}\}$**-path** is a path from s to t.

We can define the **length of a path** $\boldsymbol{P}$ as,

$$l(P) = \sum_{i=1}^{k} l(e_i)$$

SHORTEST PATH PROBLEM

From now, let us suppose that the **weight** function l is **non-negative**:

$$l(e) \geq 0 \quad \forall e \in E$$

Now, we can define a notion of distance on V:

For two vertices s and t,

$$d(s,t) = \min\{\, l(P) | P \; is \; \{s,t\} - path \,\}$$

Remark: from a strickly mathematical point of view, $\boldsymbol{d}$ **is not a distance**.

Question: what condition should we add to be sure that d is a distance?

→ Shortest path problem:

Given **two vertices** s and t, find a $\{s,t\}$-path $\boldsymbol{P}$ such that:

$$\boldsymbol{l(P) = d(s,t)}$$

In other words, that P is the **minimal length path between $\boldsymbol{s}$ and $\boldsymbol{t}$**.

SHORTEST PATH PROBLEM

Given a (**directed**) **weighted graph** $G = (V, E, l)$ with **non-negative weightes** and two nodes s and t, we have to find the **shortest path** from s to t.

→ The **Dijkstra Algorithms** [2](Dijkstra 1959) solves the problem

Intuition:

Explore the graph by **growing a ball** of nodes with **increasing distances** from the source node s

Remark: In a more general case (weights cas be negative), other theories and algorithms exists (**Belleman's equation** [4])

SHORTEST PATH PROBLEM

- Maintain a set S of vertices to which we have found the shortest path

- Maintain a distance label $L(v)$ for each vertice that define the best distance we found so far.

Dijkstra Algorithms:

(Step 0) $S = \emptyset, \quad L(s) = 0, \quad L(v) = \infty \; \forall v \in V \backslash \{s\}$

(Step 1) Chose next vertice to expand:

- Chose $v \in V \backslash S$ which minimizes $L(v)$ (use **priority queue**)
- Update $S = S \cup \{v\}$

(Step 2) Expand v:

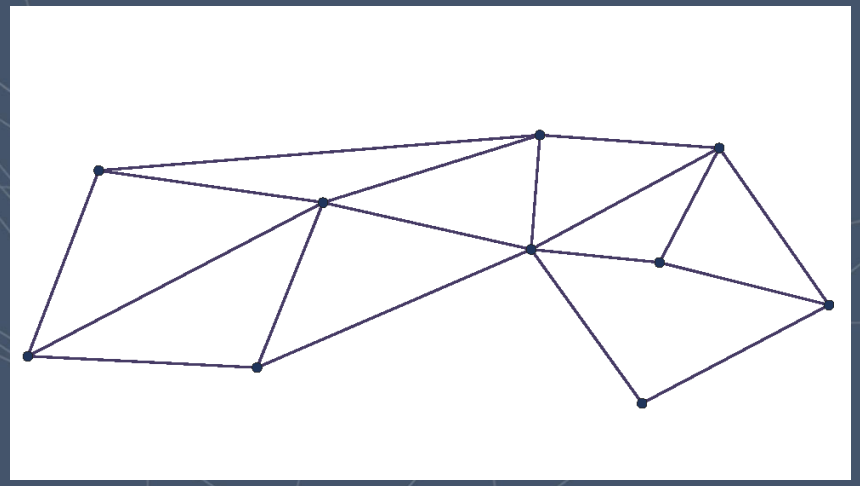
- For each edge from (v, v') with $v' \in V \backslash S$, if $L(v') > L(v) + l(v, v')$:

  Update distance: $L(v') = L(v) + l(v, v')$

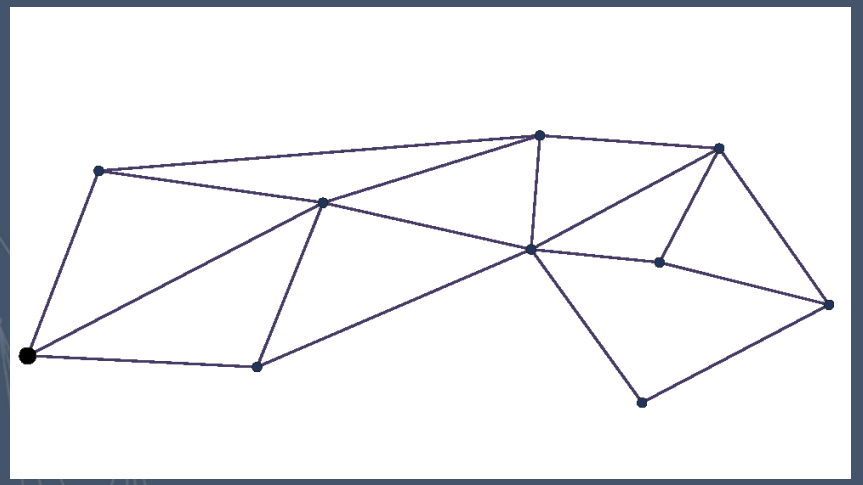
- Stop if t $\in S$

SHORTEST PATH PROBLEM

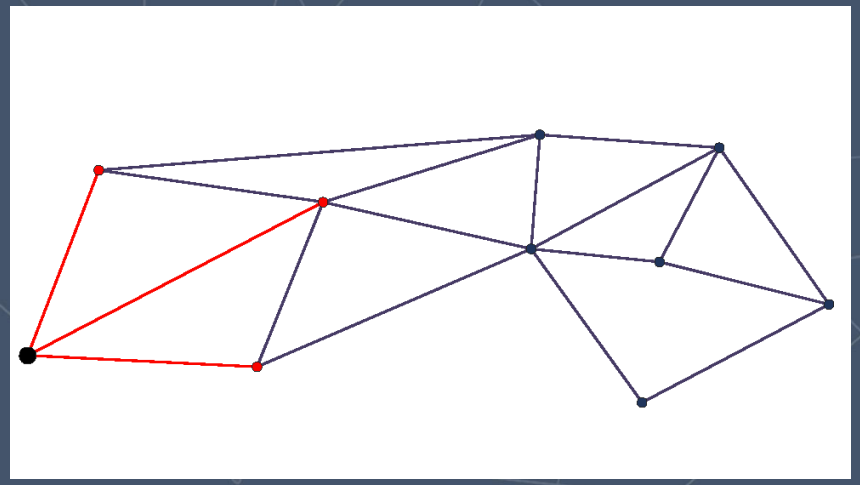
(0) Initial graph



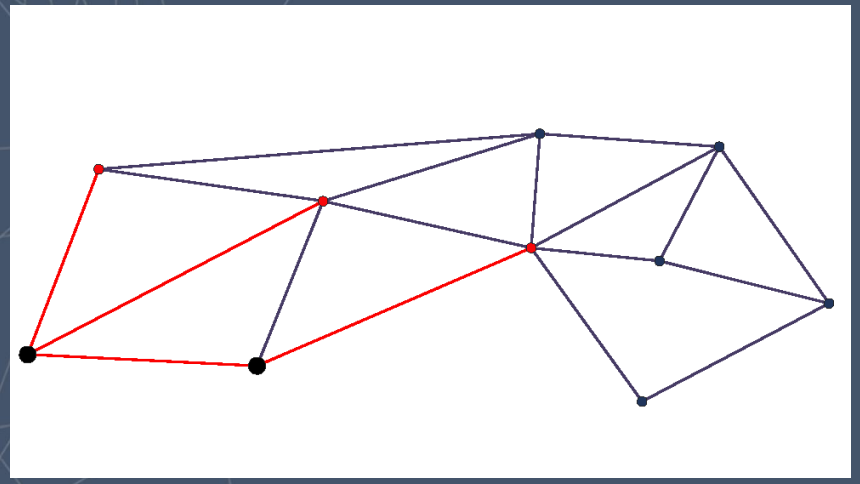
(1) Take the initial vertice



(2) Expand its edges

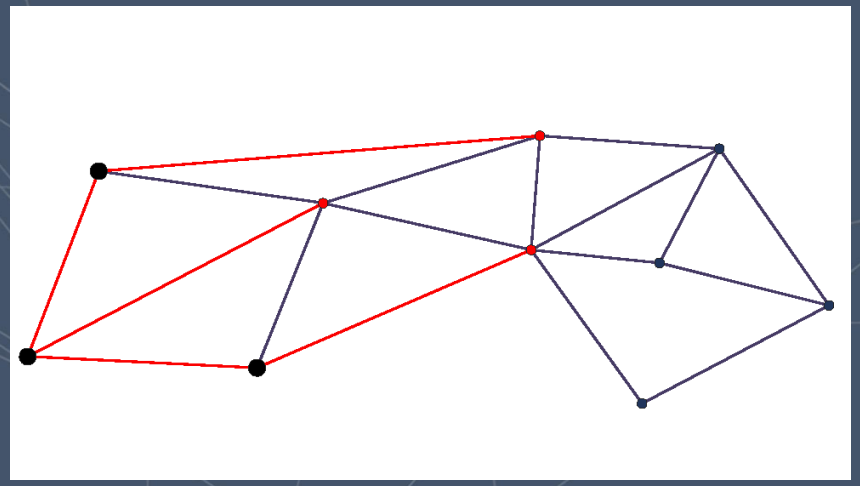


(3) Select next vertice and expand its edges

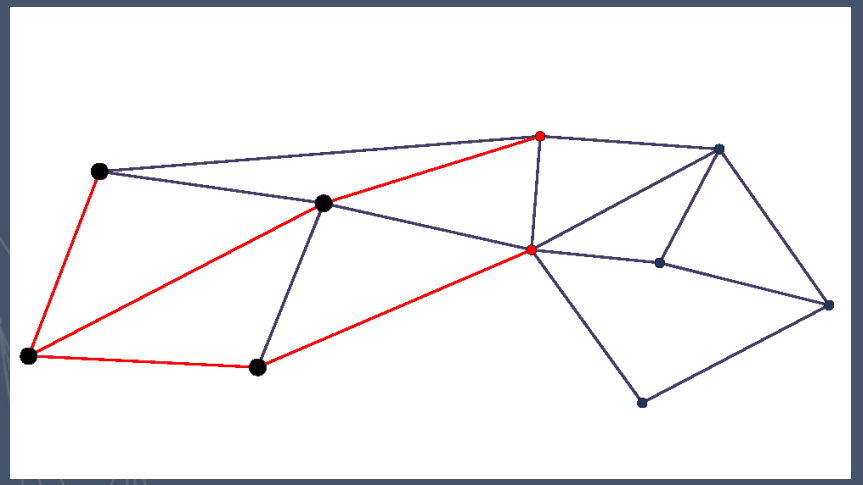


SHORTEST PATH PROBLEM

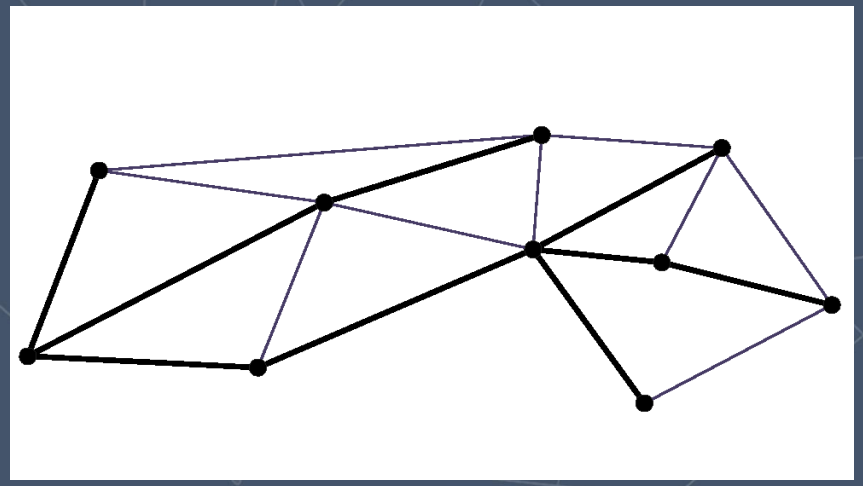
(4) Select next vertice and expand its edges



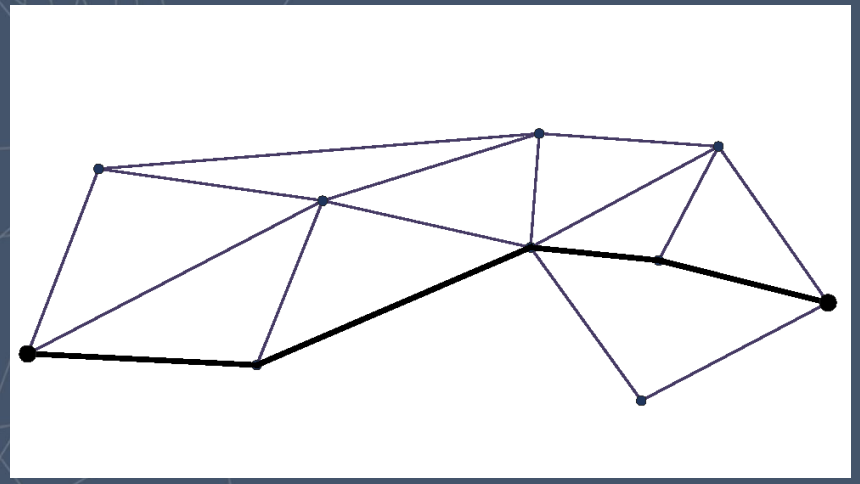
(5) Select next vertice and expand its edges



(6) Stree of all shortest paths



(7) Backtrack the shortest path



SHORTEST PATH PROBLEM

Idea of the proof [3]:

- Considera **subset S of V** such that $s \in S$ and the **labels** $L(v)$ on S are the **length of the shortest path from s**.

- Moreover, the **labels** $L(v)$ on $v \in S' = V \backslash S$ are the length of the **shortest path** from s with the **condition** that **all vertices of the path are in S** except v itself.

- Then, if we take $v \in S'$ with the **smallest label $L(v)$,** the set $S \cup \{v\}$ still **satisfy** the wanted **condition** ($L(v)$ is the length of the shortest path from s).

- Indeed, if we consider P **the $\{s, v\}$-shortest path**, it will have a **first vertice in S'** and if this first **vertice v' have to be v** because v has the smallest label $L(v)$ on S'.

- First we take $S = \{s\}$ and we expand it **inductively** until $t \in S$.

- Since all the labels of vertices **in S corresponds to shortest paths**, we have found the shortest path from s to t.

Question: why we need to suppose non-negative weights?

SHORTEST PATH PROBLEM

Since 1959, there were a lot of **variations**, **generalisations** and **speed-ups** for the Dijkstra algorithm:

For example:

- Find the k-**shortest path** between two nodes (Yen's algorithm) [5]
- Find shortest path between **all pairs of nodes** (Floyd-Warshall algorithm)
- Various **optimisation methods** to speed-up the computation time of the algorithm

Problem: How can we solve a path search problem that is:

1) **Intermodal**
2) **User-adapted**

INTERMODAL PATH SEARCH

Data structure

How to build a model of the transports network of a city that supports intermodal and user-adapted path-search?

- The data structure is a **directed** graph $G = (V, E)$ which is **embedded** in $\mathbb{R}^2$ (latitude and longitude).

- **Roads** $\rightarrow$ **Edgegs** **Crosses** $\rightarrow$ **Vertices**

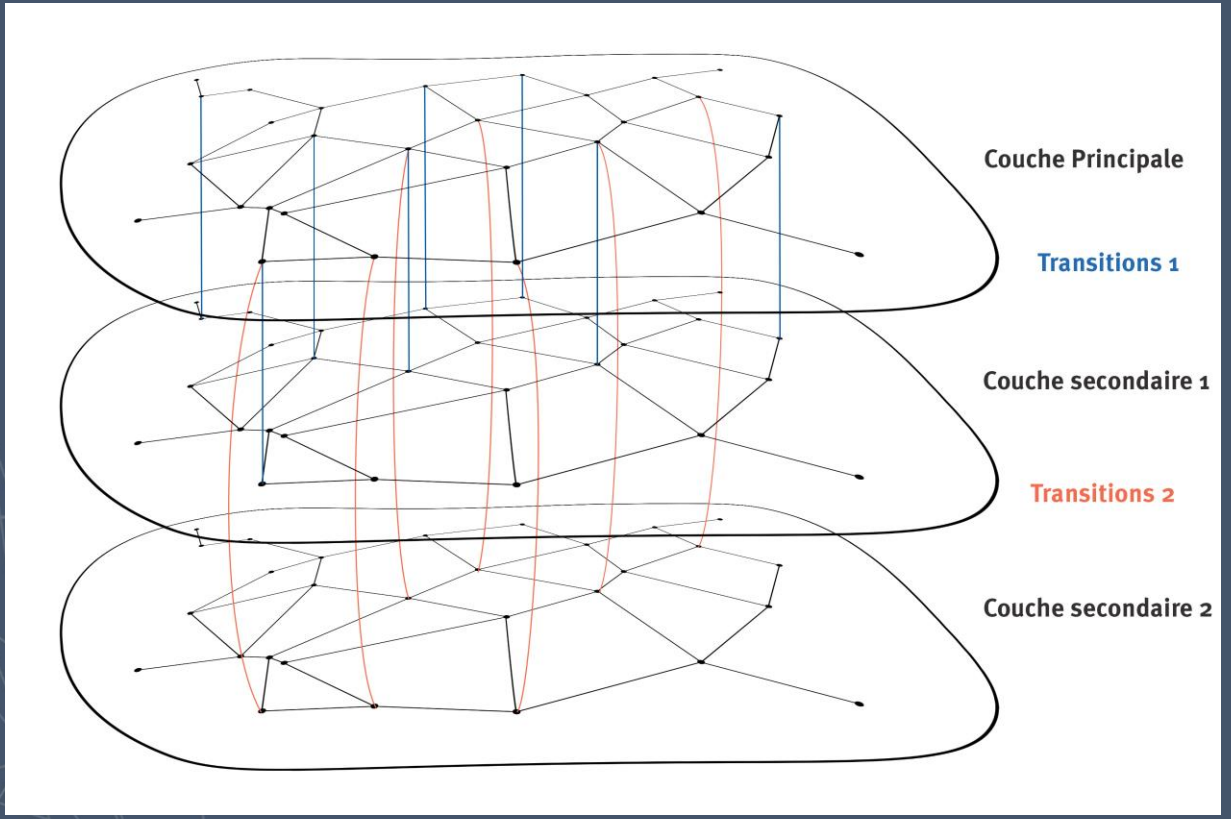
- Each edge e contain a **vector of parameters** (length, slope, maximal speed in car,…) instead of a unique weight

INTERMODAL PATH SEARCH

Data structure

The graph is organized by **layers** which represents **different types of transports:**

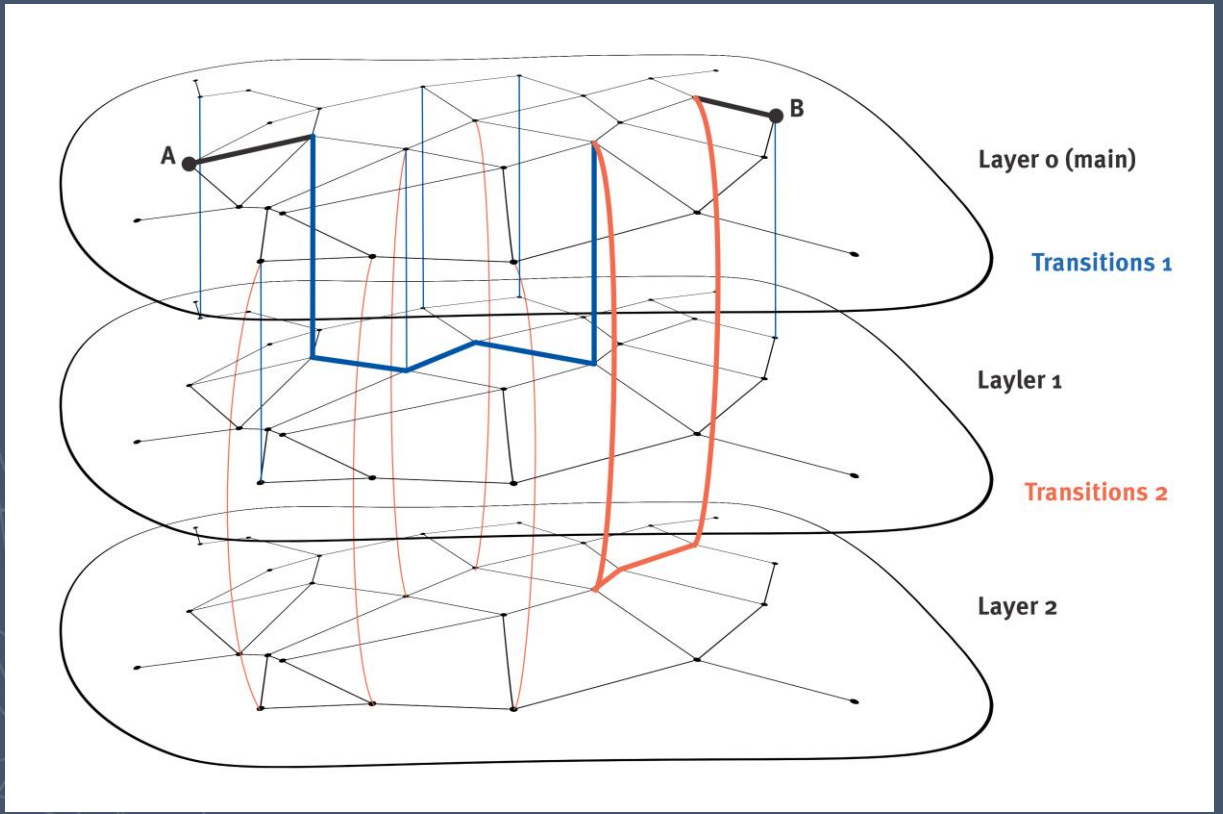
- Each **node** is included in **one layer**
- Two types of edges:
  - **transition** edges
  - **internal** edges
- But several mobility agent can share the same layer



INTERMODAL GRAPH

Data structure

- The **main layer** corresponds to ways by **foot**
- All transition links go **from the main layer** to another layer or converly
- Each path **starts** and **ends** in the main layer



INTERMODAL GRAPH

Data aquisition

Layers data (vertices and internal edges):

1) **OpenStreepMap** with the API **Overpass Turbo** (https://overpass-turbo.eu)

- OSM has tree types of objects: **nodes**, **ways** and **relations**
- Objects have **parent/children relations** between them
- Objects has **tags** organized in **key/values**

→ Export all **ways** with **highway tag** and their **children nodes** in a given region

2) **Local Public database:** Uribs (http://urbisdownload.gis.irisnet.be)

- Some data is more accurate but do not have efficient API

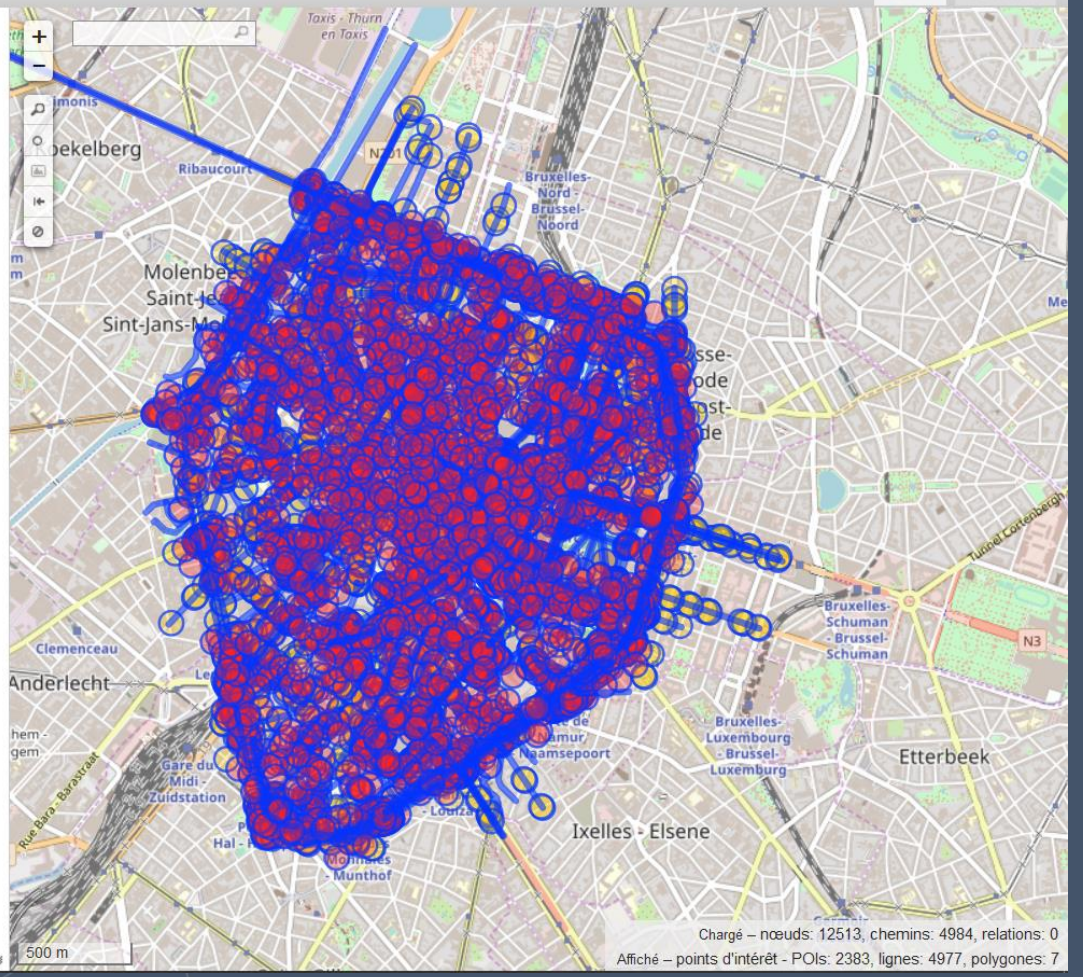
INTERMODAL GRAPH

Data aquisition: OSM Overpass Turbo, example of query

Exécuter | Partager | Exporter | Assistant | Enregistrer | Charger | Paramètres | Aide — overpass turbo

Carte | Données

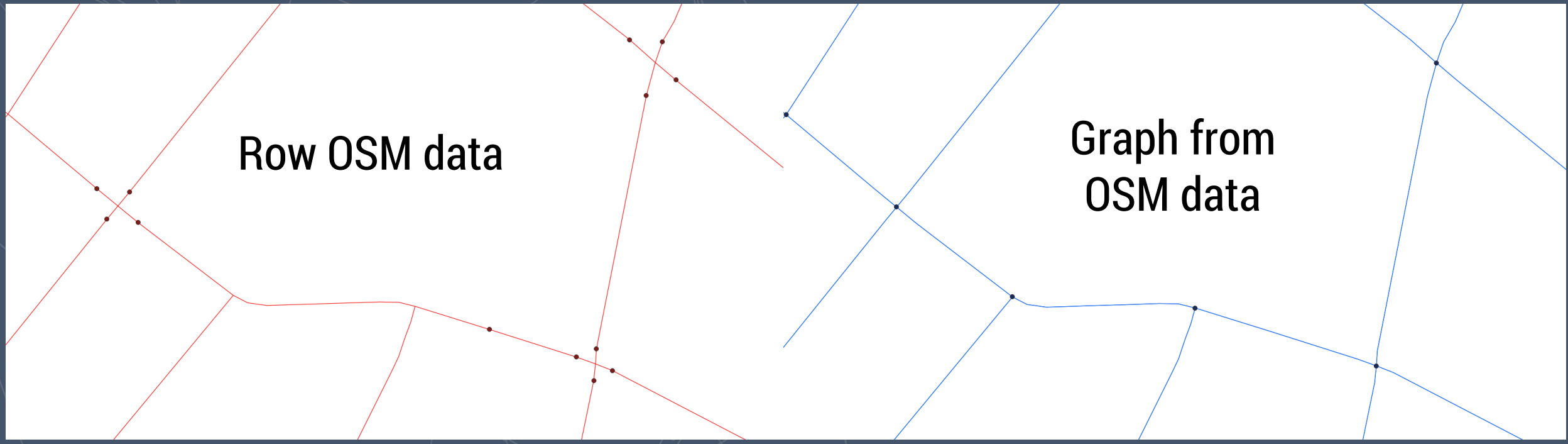
```
// STEP 0: Initial Settings --------------------------------
// 0.1) Define output format
// 0.2) Maximal time of execution
// 0.3) Rectange area (lat/lng) where the query is performed

[out:json][timeout:180][bbox:50.832,4.335,50.86,4.370];

// STEP 1: find all ways in the Region ---------------------
// 1.1) Find all 'way' objects with the tag 'highway' of any valye
// except the one that have the tag 'area'
// inside the polygone defined due to its lat/lng coordinates
// 1.2) Stock all these objets in an object .myWays

(
    way[highway][!area](poly: " 50.8329091 4.3428105 50.8499552 4.3358598 50.8591947
4.3458453 50.8531387 4.3688681 50.8470798 4.3698975 50.8403232 4.3671954 50.8326145
4.3487337");
)->.myWays;

// STEP 2: expand their child nodes ------------------------
// Remark: way objects are not located in space if we do not refer to the nodes that
they pocess
// 2.1) Recall the object .myWays
// 2.2) Access all of its children objects that are of type 'node'
// 2.3) Stock them in the object .myNodes

.myWays; node(w)->.myNodes;


// STEP 3: Take the union ----------------------------------
// 3.1) Take the union of .myWays and .myNodes
// 3.2) Save it in the object .all

( .ways; .nodes;)->.all;

// STEP 4: Print/export the data ---------------------------
// 4.1) Export the object .all
// 4.2) Precise how detailed is the exported data with 'meta'

.all out meta;
```



INTERMODAL GRAPH

Data aquisition

OSM data is **not** organized as a **graph** => need **preprocessing** algorithms



INTERMODAL GRAPH

Data aquisition

Transition between layers (transition edges):

Data from each **actor of transports**

- Some data is on OSM or Urbis (as the Villo stops)
- Use different API for each actor of transports (open or private)

INTERMODAL GRAPH

Data aquisition

Reunite all data in a **unique multi-layer graph** structure

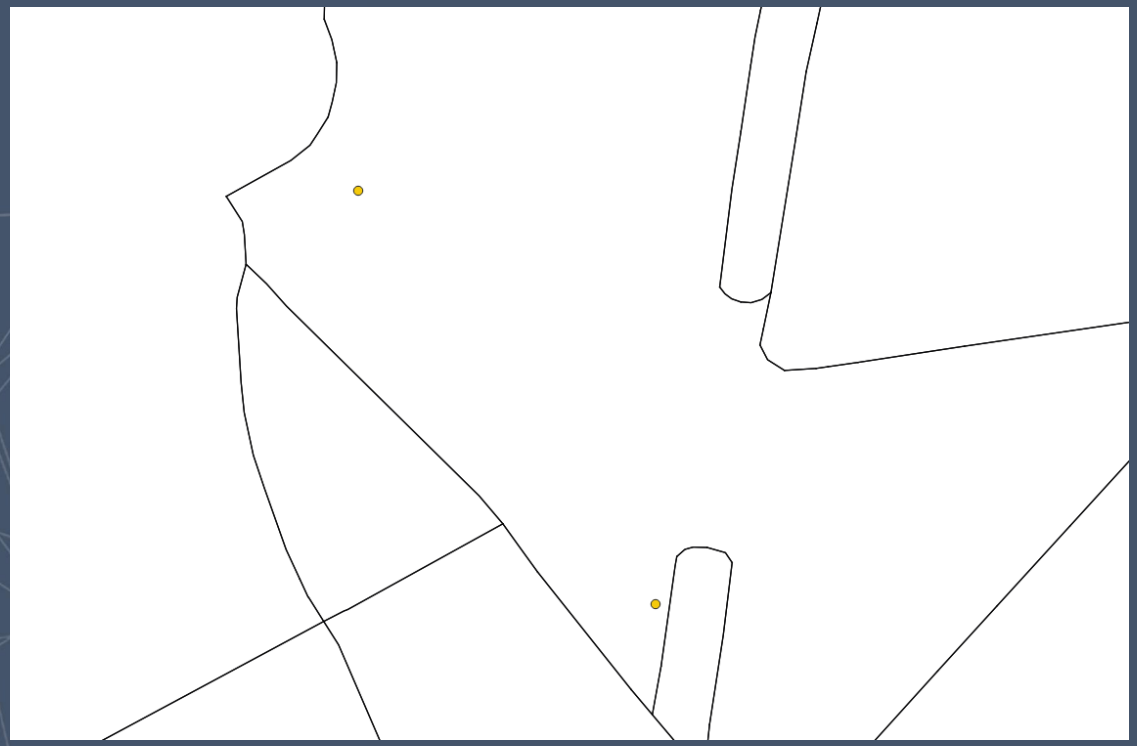
- All informations are stocked in the **parameter vector** of **edges**

Challenge:

How to **reunite** the data in the best/more realistic way?

How to **link a Villo station** to the rest of the **network**?

→ need algorithms of **computations geometry**



INTERMODAL GRAPH

Algorithm

Use any classical shortest path algorithms (any variation of the Dijkstra algorithms) with some additional details:

The **cost** of a edge is not a **unique** constant **weight**, but is defined by a **cost funciton** depending on the **parameters of the link**, the **layer**, the **user**, the **environmental** condition and the parameters of the **search**.

- This allows us to find shortest paths **depending on**,
  - **User's** preferences and possibilites
  - **Environmental** conditions

- This also **allows** us to:
  - Find **several shortest paths** that use different modalities
  - Find shortest path with different balances of **time**, **cost** and **ecological impact**

INTERMODAL GRAPH

Challenges:

1) How the measure the **« real cost »** of a road
    - **Theoretical model**
    - **Learning**
2) How to make the graph **geometrically accurate?**
    - Data from different sources may be slightly different
    - Some **data** may be **missed**
    - All the reality of a road network can not be stored in data bases
3) How to **optimize** this kind of algorithms on this type of graph **topology**?

OPTIMIZATION METHODS

Heap

To **choose the next vertice** to expand, the Dijkstra algorithm have to take the vertice with the **minimal distance** tag $L(v)$ which is a **priority queue** that is constantly updated and may become very long.

The use of **heap** (**binary** or **Fibonacci** [6], [7]) can largely **reduce the computation time** of the algorithms.

Complexity:

Dijkstra with list: $O(V^2)$

Dijkstra with binary heap: $O(\,V log_2(V)\,)$

Idea:

To extract the minimum value of a prority queue,

we only have to **partially order the queue** (not totally)

data structure	insert	remove maximum
ordered array	N	1
unordered array	1	N
heap	$\log N$	$\log N$
impossible	1	1

Order of growth of worst-case running time for priority-queue implementations

Image from [7]

OPTIMIZATION METHODS

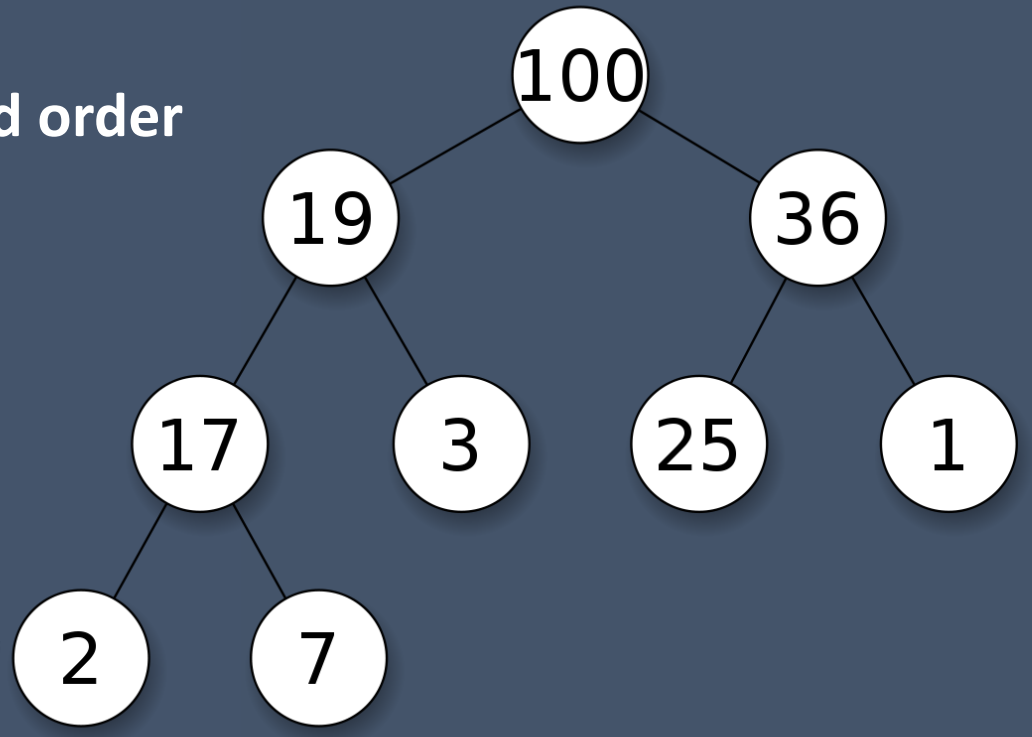
Binary heap:

A structure of binary tree where each **childred** is **graeter or equal** that its **parent**.

So the **largest element** is found on the **root** of the tree.

Remark: the structure works as well with the **reversed order**

largest => **smaller**



OPTIMIZATION METHODS

Binary heap:

The binary heap structure car be stored in an **array**.

For that we store the elements starting at 1 represented in **depth-level order** in the array.

This allow easy **access to parents and children**:

Children of element at position p: $2p$ and $2p + 1$

Parrent of element at position p: $floor(^{p}/_{2})$

Exercice: prove it

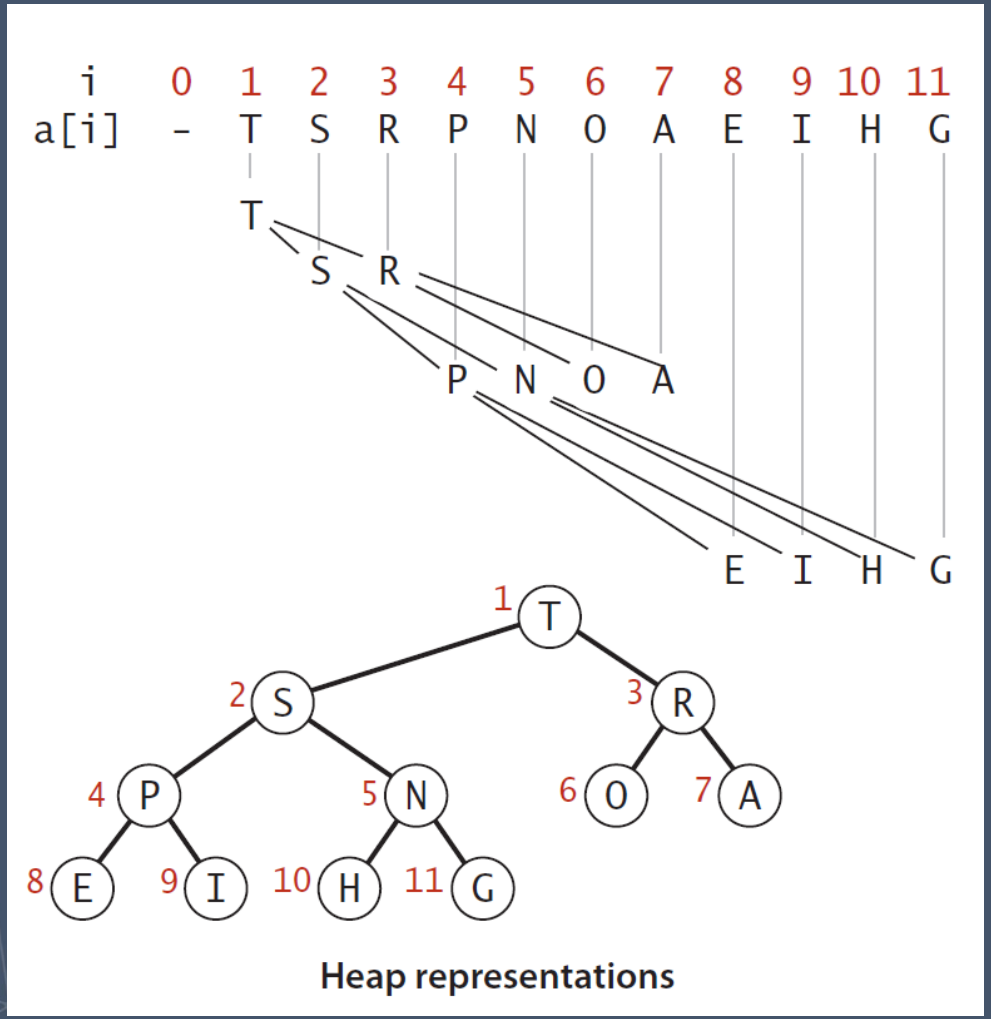


Image from [7]

OPTIMIZATION METHODS

Binary heap: Preliminary operations

- Swim:

When an element violates the heap order because it is **larger than its parent** (it is too **low** in **depth-level** of the tree).

- **Exchange** it with its **parent** element
- **Repeat** until the heap **order** is **restored**

- Sink:

When an element violates the heap order because it is **smaller than** (at least) **one of its children** (it is too **high** in **depth-level** of the tree).

- **Exchange** it with the **largest child**
- **Repeat** until the heap **order** is **restored**

Question: Why swim and sink restore the heap order and do not create other heap order violations?

OPTIMIZATION METHODS

Binary heap:

Insertion:

- **Insert** the new element in the **end** of the array
- « **Swim** » the element until the heap order is restored

Remove maximal element:

- **Remove** the **first element** of the array (**root** of the tree)
- Put the **last element** of the array on the **first place** (on the **root**)
- « **Sink** » the element until the heap order is restored

Question: See that the complexity of these operations is $O(\log(n))$ where n is the length of the queue.

Question: In Dijkstra algorithm, what to do when the distance label $L(v)$ is updated?

OPTIMIZATION METHODS

Bidirectional search ([9], [12])

- One **forward** search from the **source** node s
  → On the **original graph**
- One **reversed** search from the **targer** node t
  → On the **reversed graph**
- **Alternate** the two searches until the two searches « **meets** » at a vertice m.
- **Reconstruct the path**:
  - P_f: shortest $\{s, m\}$-path
  - P_r: shortest $\{t, m\}$-path
  - P, the shortest $\{s, t\}$-path is reconstructed as $P_f + rev(P_r)$

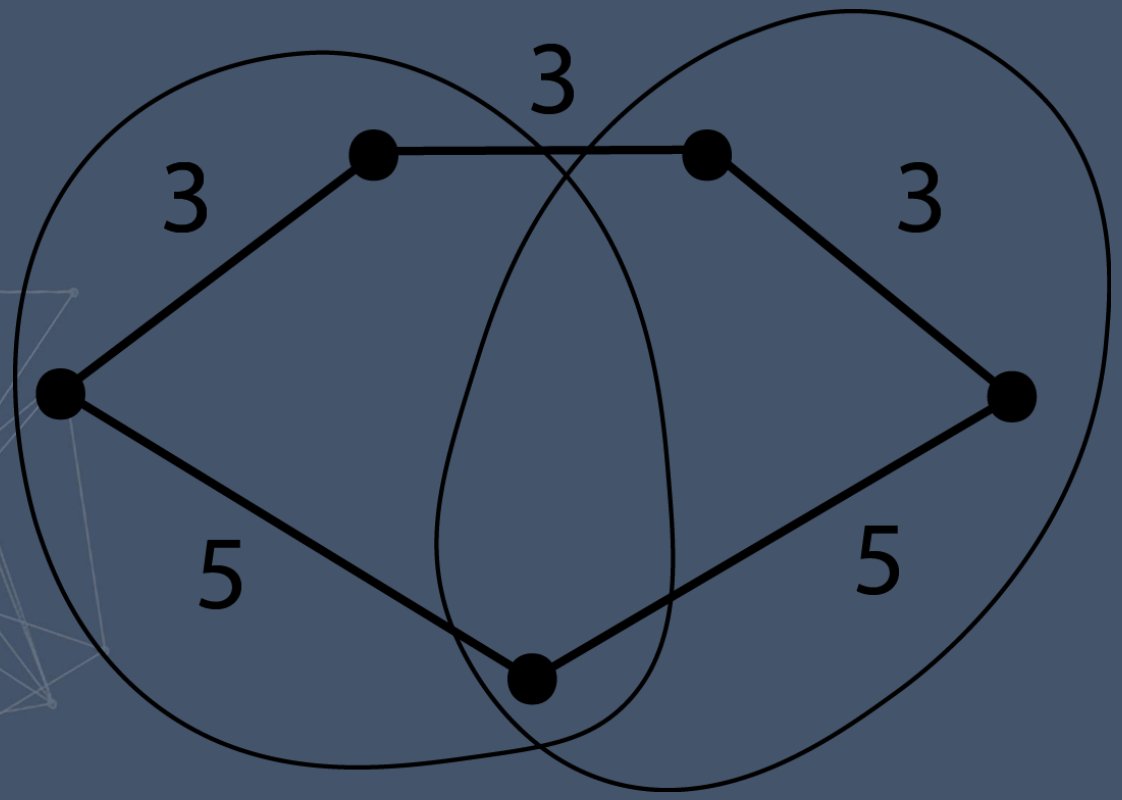
OPTIMIZATION METHODS

Bidirectional search

Problem:

When the two sets S_f and S_r of the respectively **forward** and **reversed** searches **intersects**, the **optimality is not guaranteed**.

→ Need for a **stopping condition**



OPTIMIZATION METHODS

Bidirectional search

Stopping condition:

Keep track of the **forward and reversed distance label** as $L_f(v)$ and $L_r(v)$

Define μ as the **length** of the **minimal path founded so far** as:

When an edge $e = (v, w)$ is scanned (when v is included in S_f) in the forward search and $w \in S_r$, we update μ if $L_f(v) + l(v, w) + L_r(w) < \mu$ and conversly.

Stop when $top_f + top_r \geq \mu$

Where top_f and top_r are the **minimal** value of the **priority queue** of respectively the forward and the reversed search.

OPTIMIZATION METHODS

Bidirectional search

Stopping condition: Idea of the proof

Suppose we have searched until stopping condition but there is a path P of length $\mu' < \mu$.

So there is and edge $e = (v, w)$ in P such that:

$$d(s, v) < top_f \quad \text{and} \quad d(w, t) < top_r$$

$$\text{Indeed, } \mu' = d(s, v) + l(v, w) + d(w, t) < top_f + top_r$$

This means that v is already scanned in the forward search and w is already scanned in reversed search.

So, when the second of them was expanded, the edge e was scanned and so the path P should have been found.

OPTIMIZATION METHODS

Bidirectional search

For road networks:

Search space: $D_1 \sim \pi R^2 \implies D_2 \sim 2\pi (R/2)^2$

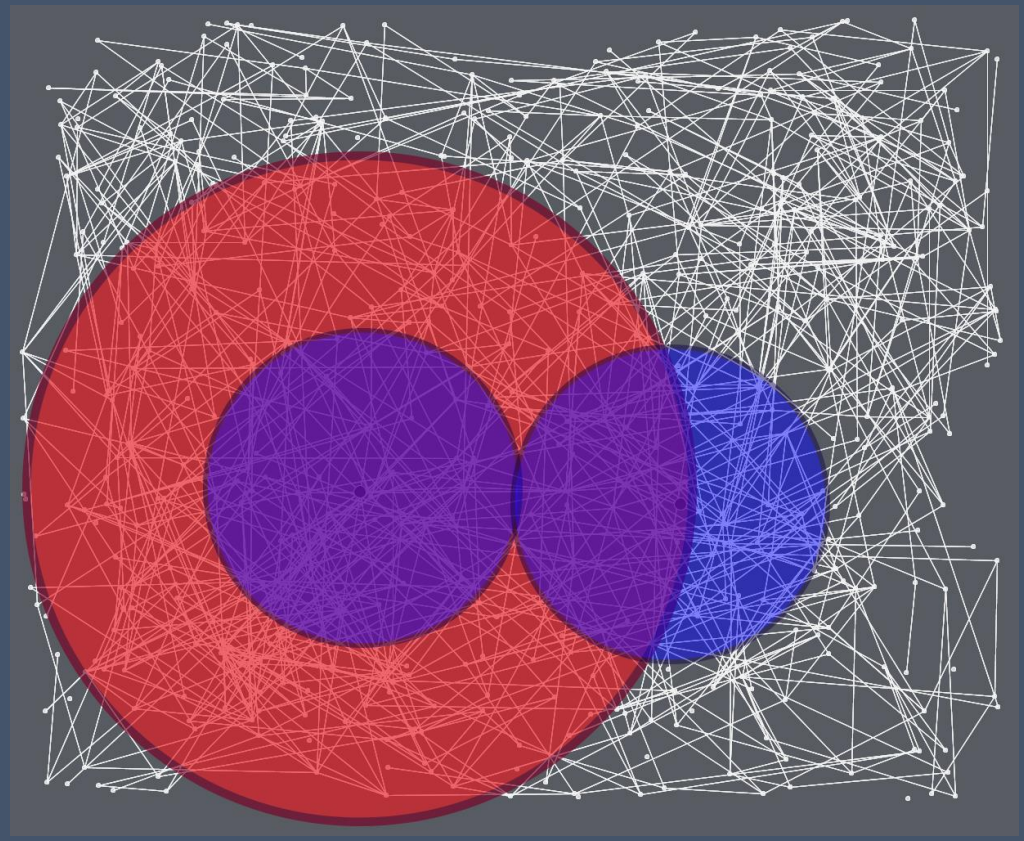
→ Decrease the search space by 2

For search with constant branching factor b:

Search space: $D_1 \sim b^d \implies D_2 \sim 2\, b^{d/2}$

→ Decrease the search space by $\frac{1}{2}\, b^{d/2}$

*The **efficiency** of the optimizatin method largely **depends** on the **topology** of the graph!*



OPTIMIZATION METHODS

The following optimization methods use **preprocessing** techniques or **external informations** about the graph.

Naive preprocessing: Compute **all shortest pathes** between all points and then just give the answer by accessing it in the memory.

Problem: in most of the **modern** applications handle **networks** are **too large** to allow it in terms of **memory** and **computational time**.

OPTIMIZATION METHODS

A* [8]

Idea:

Use **heuristic information** to determine **which vertice explore first**

Intuitively, if the vertice t is on the **right** of the vertice s, it may **not be a good idea** to **search** far **on the left of s**.

In this case, the **heuristic** can be considered as a **prediction for the distance** from a vertice v to the target vertice t.

OPTIMIZATION METHODS

A*

Consider a **potential function** $\pi: V \to \mathbb{R}$ such that $\pi(t) = 0$.

We say that π is **feasible** if:

$$\forall\, u, v \in V \quad \mathrm{d(u, v)} - \pi(u) + \pi(v) \geq 0$$

In other words, that π **never overestimates** d.

Then, **redefine** a new edge **weight**

$$l_\pi(u, v) \coloneqq l(u, v) - \pi(u) + \pi(v).$$

A* algorithm: Dijkstra algorithm on the same graph $G = (V, E)$ but with **new weights** l_π

OPTIMIZATION METHODS

A*: Idea of the proof

By the fact that the heuristic π is **feasible**, the new weighted graph $G_\pi = (E, V, l_\pi)$ has **non-negative weights** (why?), which is required for the optimality of the Dijkstra algorithm.

All the $\{s, t\}$**-paths lengths** are **changed** by the same **constant** value,

$$l_\pi(P) = l(P) - \pi(s) + \pi(t) = l(P) - \pi(s)$$

So a shortest $\{s, t\}$-path in G_π is also a **shortest path** in G.

Since the A* algorithm on G is a Dijkstra algorithm on G_π, it give the shortest path on G.

OPTIMIZATION METHODS

A*

To note:

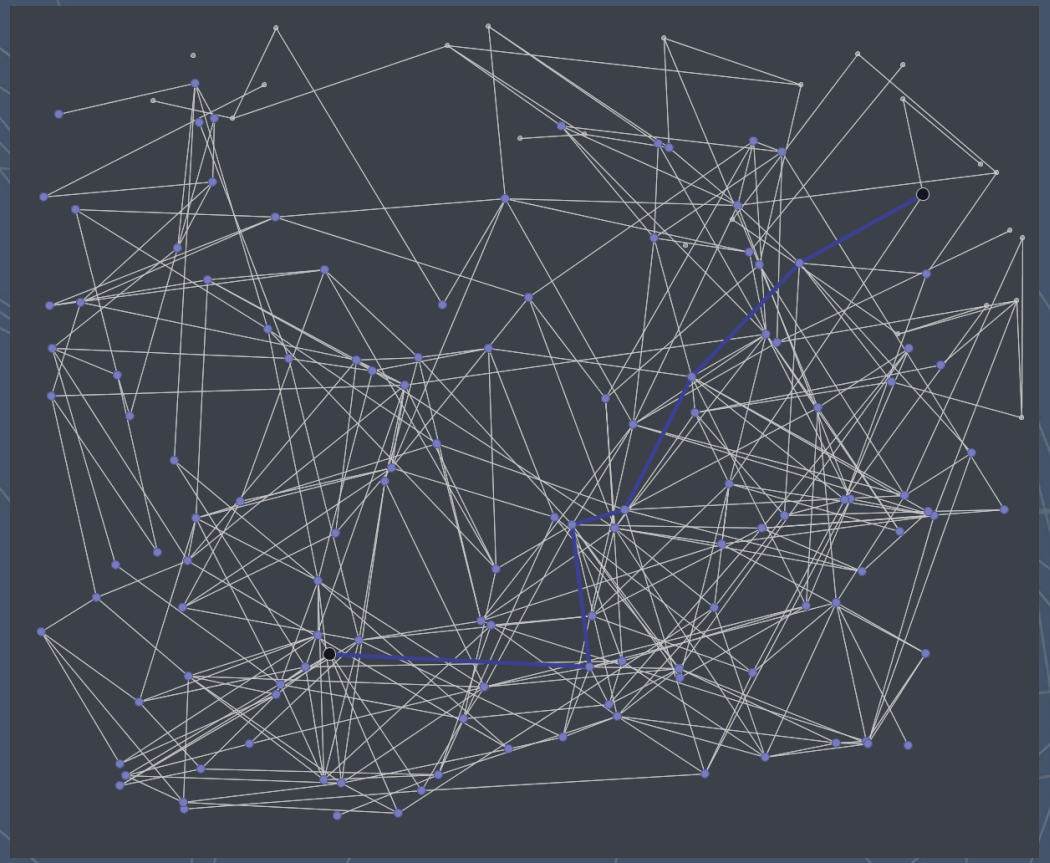
- The A* algorithm **reduce the search space** S of a run and so reduce the computational time.
- The aim of the heuristic $\pi(v)$ is to give a **lower bound** to $d(v,t)$.
- The **closer** is the **lower bound** the **better** the A* will work

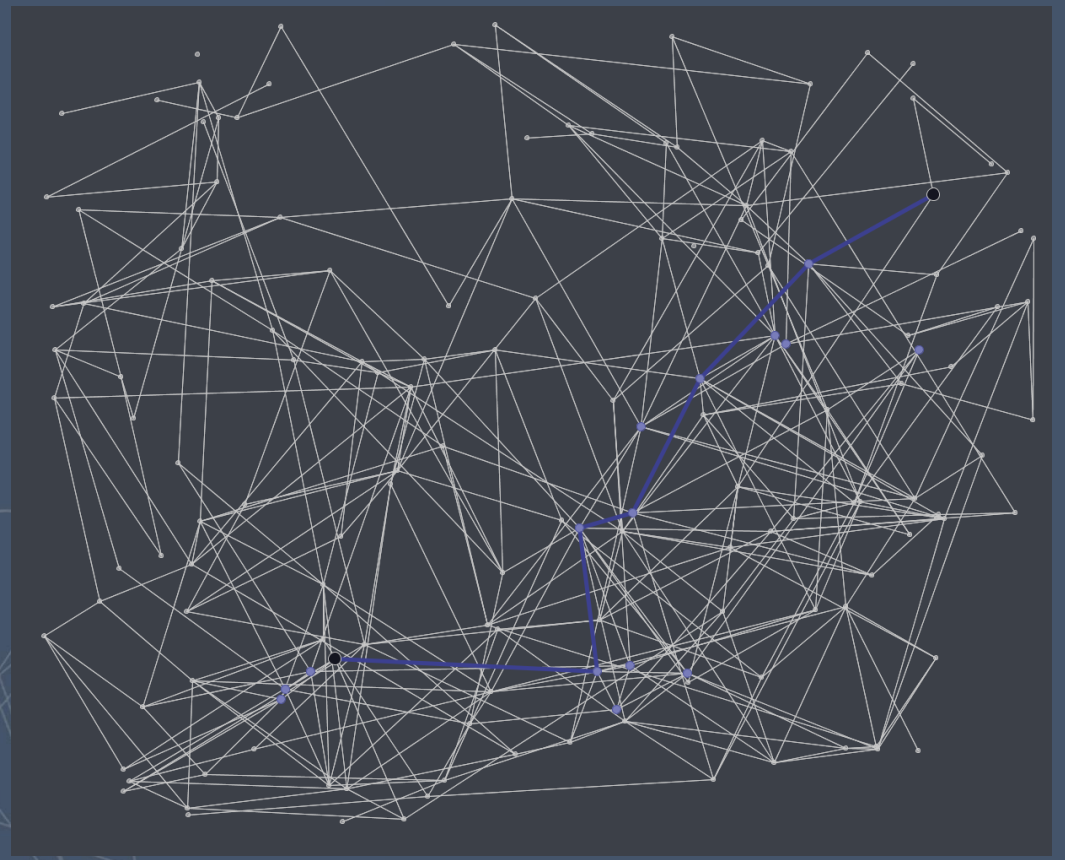
Remark:

In **roads networks** for routing algorithms, we can for example take as heuristic π the **straight-line distance** to t with the **maximal** possible **speed** on this region. (Why maximal?)

OPTIMIZATION METHODS

A* with Euclidian distance as π vs. **Dijkstra**





Dijkstra: **126** explored vertices

A*: **17** explored vertices

OPTIMIZATION METHODS

Landmarks [10]

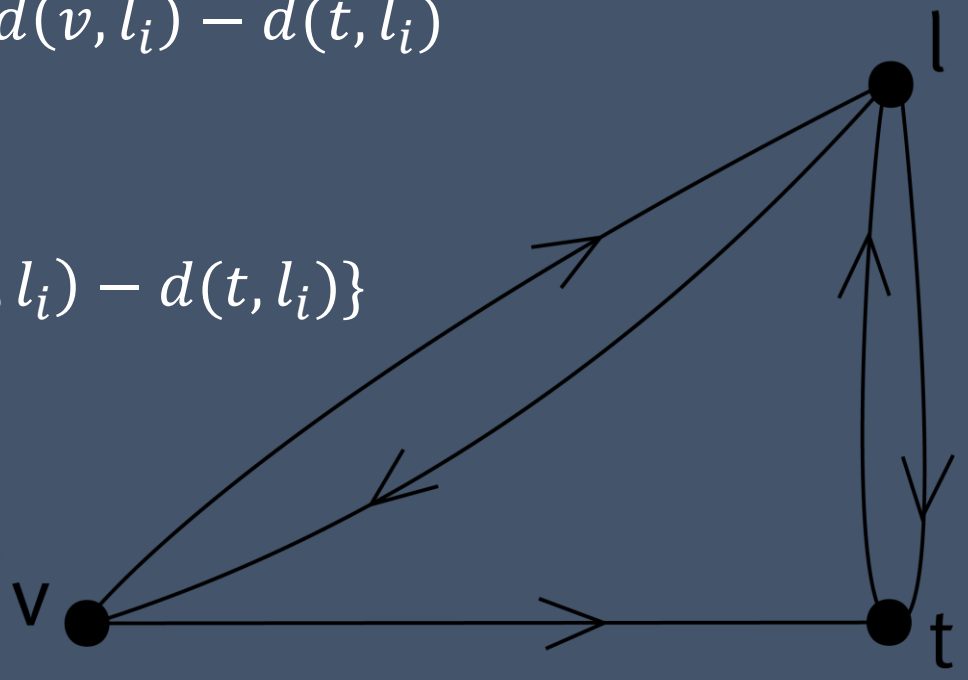
- Select k **landmarks** ($l_1, \dots, l_k$): vertices for which we compute **distances** to and from all other vertices
- For each landmark l_i, each vertice v and the target vertice t, we can compute two independent **triangle inequalities**:

$$d(v,t) \geq d(l_i,t) - d(l_i,v) \quad \text{and} \quad d(v,t) \geq d(v,l_i) - d(t,l_i)$$

- Use it as a **lower bound for the distance $d(v,t)$**:

$$d(v,t) \geq max_{i=1,\dots,k}\{d(l_i,t) - d(l_i,v),\ d(v,l_i) - d(t,l_i)\}$$

- Use this lower bound as **heuristic for A***



OPTIMIZATION METHODS

Landmarks

Landmarks can **considerably reduce the search sapce** S.

The **better** is the **lower bound** (for $d(v,t)$), the **more** they it **reduce the search space**.

→ We have to found landmarks that give good lower bounds.

At one **run**, select only a **few landmarks** and not all, otherwise the computations of the maximal bound will be too long.

Non-trivial question: how to **find** the **best possible landmarks** (the one who reduce in average the most the search space)

OPTIMIZATION METHODS

There are many others optimization methods for shortest path problem and it remains an active field of research.

Two things to note:

1. Their **efficiency** depends on the **topology** of the graph
2. They often require **preprocessing**

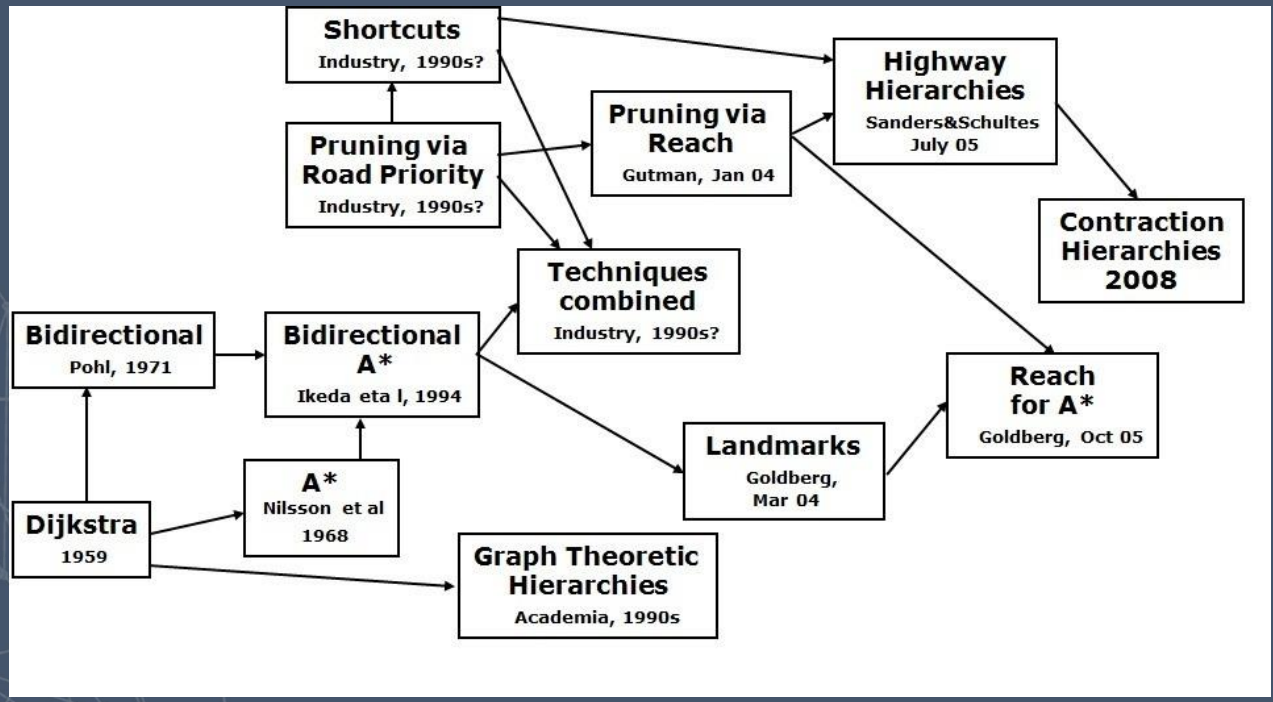


Image from [11]

BACK TO INTERMODAL GRAPH

Problem:

- Speed-up techniques require **preprocessing**
- The graph is **intermodal** and **user-adapted**

→ **Preprocessing is very challenging** (different users have different shortest paths)

How can we make preprocessing for each user and in intermodal graph?

- **Several preprocessings**?
- **Generalize preprocessing** that keep flexible information?

FURTHER QUESTIONS AND RESEARCH

1) **Which** speed-up use?

- Their **efficiency** depends on the **topology** of the graph
- The intermodal graph has its **specific topology**

→ **Which speed-up technique** is the better to use ?

2) **How** to use it?

- **Preprocessing** is very **challenging**

→ How to **adapt current speed-up** techniques to intermodal graph ?

THANK YOU

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