# Maximum $st$-flow in directed planar graphs via shortest paths

Glencora Borradaile[1] and Anna Harutyunyan[2,⋆]

[1] Oregon State University
[2] Vrije Universiteit Brussel

**Abstract.** In this paper, we give a correspondence between maximum flows and shortest paths via duality in *directed* planar graphs with no constraints on the source and sink.

## 1 Introduction

The asymptotically best algorithm for max $st$-flow in directed planar graphs is the $O(n \log n)$-time *leftmost-paths algorithm* due to Borradaile and Klein [4], a generalization of the seminal *uppermost-paths algorithm* by Ford and Fulkerson for the $st$-planar case [6]. Both algorithms augment $O(n)$ paths, with each augmentation implemented in $O(\log n)$ time. However, this bound is achieved using the overly versatile dynamic-trees data structure [11] in Borradaile and Klein's algorithm, and only priority queues in the $st$-planar case [7]; priority queues are arguably simpler and more practical than dynamic trees [13]. The reason priority queues are sufficient for the $st$-planar case is due the equivalence between flow and shortest-paths problems via duality. We conjecture that an augmenting-paths algorithm for the more general problem can be implemented via $O(n)$ priority-queue operations. We make progress toward this goal by showing that maximum flow in the general case is equivalent to computing shortest paths in a covering graph, even though, algorithmically, we do not improve on the algorithm of Johnson and Venkatesan [9].

**Background** For omitted proofs and full definitions and an additional shortest-path based algorithm, please see the full version of this paper [2]. For a full background on planarity and flow, see Refs. 1 and 10.

For a path $P$, left$(P)$ (right$(P)$) is the maximal subset of the darts whose head or tail (but not both) is in $P$, and who enter or leave $P$ from the left (right). We define the graph $G \not\!\!\!\!\times P$ as the graph *cut open along* $P$. $G \not\!\!\!\!\times P$ contains two copies of $P$, $P_L$ and $P_R$, so that the edges in left$(P)$ are adjacent to $P_R$ and the edges in right$(P)$ are adjacent to $P_L$. The parameter $\phi$ is the minimum number of faces that any $s$-to-$t$ curve must pass through [10].

An *excess (deficit)* vertex is that with more flow entering (leaving) than leaving (entering). A flow is *maximum* if and only if there are no residual source-to-sink paths [6]. A pseudoflow is *maximum* if and only if there are no residual

⋆ Work done while at Oregon State University.

paths from a source or excess vertex to a deficit or sink vertex [8]. The notion of *leftmost* flows is essential in understanding this paper, and is covered in Ref. 1; a flow is leftmost if it admits no clockwise residual cycles. The following lemma follows directly from definitions of leftmost and the following theorem provides structural insight into leftmost paths and flows.

**Lemma 1.** *A leftmost circulation can be decomposed into a set of flow-carrying clockwise simple cycles.*

**Theorem 1.** *Let $L$ be the leftmost residual $s$-to-$t$-path in $G$ w.r.t. c.w. acyclic capacities $\mathbf{c}$. Let $\mathbf{f}$ be any $st$-flow (of any value). Then no simple $s$-to-$t$ flow path crosses $L$ from the left to right.*

**Infinite covers** Embed $G$ on a sphere and remove the interiors of $f_t$ and $f_s$. The resulting surface is a cylinder with $t$ and $s$ embedded on opposite ends. The repeated drawing of $G$ on the universal cover of this cylinder [12] defines a *covering graph*[3] $\mathcal{G}$ of $G$. For a subgraph $X$ of $G$, we denote the subgraph of $\mathcal{G}$ whose vertices and darts map to $X$ by $\mathcal{G}[X]$. We say $\bar{X} \subset \mathcal{G}[X]$ is a *copy* of $X$ if $\bar{X}$ maps bijectively to $X$. We number the copies of a vertex $u$ from left to right: $\mathcal{G}[u] = \{\dots u^{-1}, u^0, u^1, \dots\}$, picking $u^0$ arbitrarily. We say that $\bar{X}$ is an *isomorphic* copy of $X$ if $\bar{X}$ is isomorphic to $X$.[4] For a simple $s$-to-$t$ path $P$ in $G$, we denote by $P^i$ the isomorphic copy of $P$ in $\mathcal{G}[P]$ that ends at $t^i$. $G^i_P \cup P^{i+1}$ is the finite component of subgraph of $\mathcal{G} \text{✂} P^i \text{✂} P^{i+1}$. Lemma 2 relates clockwise cycles in $\mathcal{G}$ and $G$ and Lemma 3 is key to bounding the size of the finite portion of $\mathcal{G}$ required by our algorithm:

**Lemma 2.** *The mapping of the following into $G$ contains a clockwise cycle: Any $u^i$-to-$u^j$ path in $\mathcal{G}$ for $u^i \in G^i_P$, $u^j \in G^j_P$, and $i < j$ and any simple clockwise cycle in $\mathcal{G}$.*

**Lemma 3 (Pigeonhole).** *Let $P$ be a simple path in $G$. Then $\bar{P}$ contains a dart of at most $\phi + 2$ copies of $G$ in $\mathcal{G}$. If $P$ may only use $s$ and $t$ as endpoints, then $\bar{P}$ contains darts in at most $\phi$ copies of $G$ in $\mathcal{G}$.*

## 2 Maximum flow, shortest paths equivalences

Starting with c.w. acyclic capacities $\mathbf{c}$, we compute the leftmost maximum flow in a finite portion of $\mathcal{G}$ (containing $k$ copies of $G$), $\mathcal{G}_k$, via embedding two additional vertices $T$ (above) and $S$ (below) the cover, and computing the leftmost max $ST$-flow $\mathbf{f}^{ST}$ via the priority-queue implementation of Ford and Fulkerson's uppermost-paths algorithm (Section 2.1). Note that shortest-paths/priority-queue based implementations always produce leftmost flows. From $\mathbf{f}^{ST}$ we show how to extract the *value* of max $st$-flow $|\mathbf{f}|$ (Section 2.2). Using this value we are able to modify the capacities $\mathbf{c}^{ST}$ in a way that allows to extract $\mathbf{f}$ from $\mathbf{f}^{ST}$ (Section 2.3). While this method requires a factor $k$ additional space, we believe this can be overcome in future work (Section 2.4).

[3] This is similar to a cover used by Erickson analysis [5]; we remain in the primal.
[4] Note that an isomorphic copy need not exist, e.g., the boundary of $f_s$.

### 2.1 The finite cover

Let $L$ be the leftmost residual $s$-to-$t$ path in $G$ and let $\mathbf{f}$ be the (acyclic) leftmost maximum $st$-flow in $G$. Let $\mathcal{G}_k$ be the finite component of $\mathcal{G} \text{✂} L^0 \text{✂} L^k$, made of $k$ copies of $G$ (plus an extra copy of $L$). We start by relating a max multi-source, multi-sink maximum flow in $\mathcal{G}_k$ ($\mathbf{f}_1$) to a max pseudoflow $\mathbf{f}_0$ (Lemma 4), and, in turn relate $\mathbf{f}_0$ to $\mathbf{f}$ (Lemma 5). This will illustrate that the flow in the central copy of $\mathcal{G}_k$ is exactly $\mathbf{f}$. However, $\mathbf{f}_1$ is not necessarily leftmost and so, we cannot necessarily compute it. We relate $\mathbf{f}_1$ to the leftmost max $ST$-flow in $\mathcal{G}_k^{ST}$ (which we can compute), in Section 2.2, from which we can compute the *value* of $\mathbf{f}$.

Let $\mathbf{f}_0$ be a flow assignment for $\mathcal{G}_k$ given by $\mathbf{f}_0[\bar{d}] = \mathbf{f}[d]$, $\forall \bar{d} \in \mathcal{G}[d]$. We overload $\mathbf{c}$ to represent capacities in both $G$ and $\mathcal{G}_k$, where capacities in $\mathcal{G}_k$ are inherited from $G$ in the natural way. In $\mathcal{G}_k$ and $G$, we use residual to mean w.r.t. $\mathbf{c}_{\mathbf{f}_0}$ and $\mathbf{c}_{\mathbf{f}}$, respectively.

**Lemma 4.** *For $k > \phi + 2$, $\mathbf{f}_0$ is a maximum pseudoflow with excess vertices on $L^0$ and deficit vertices on $L^k$.*

*Proof.* Since $\mathbf{f}_0$ is balanced for all vertices in $\mathcal{G}_k$ except those on $L^0$ and $L^k$ and it follows from Theorem 1 that $V^+$ resp. $V^-$ belong to $L^0$ resp. $L^k$, where $V^+$ resp. $V^-$ denote the set of excess resp. deficit vertices. Since a source-to-sink path in $\mathcal{G}_k$ maps to an $s$-to-$t$ path in $G$, it remains to show that there are no $V^+$-to-$T$, $S$-to-$V^-$ or $V^+$-to-$V^-$ residual paths. By the Pigeonhole Lemma and Part 2 of Lemma 2, the last case implies a clockwise residual cycle in $G$, contradicting $\mathbf{f}$ being leftmost. The first two cases are similar, we only prove the first case here.

Consider the flow assignment for $\mathcal{G}$: $\mathbf{f}'[\bar{d}] = \mathbf{f}[d]$, $\forall \bar{d} \in \mathcal{G}[d]$. For $v^+ \in V^+$ to be an excess vertex, there must be a $v$-to-$t$ flow path $Q$ in $\mathbf{f}$ where $v$ is the vertex in $G$ that $v^+$ maps to. There is a copy $\bar{Q}$ of $Q$ in $\mathcal{G}$ that starts at $v^+$, and by Theorem 1 is left of $L^0$. For a contradiction, let $R$ be a $v^+$-to-$t^i$ residual path, for some $t^i \in T$. Then, $rev(Q) \circ R$ is a residual $t^j$-to-$t^i$ path in $\mathcal{G}$ (w.r.t. $\mathbf{f}'$), $j \leq i$. If $j = 0$, $\bar{Q} \circ rev(L^0[v^+, t^0])$ is a clockwise cycle, which, by Part 2 of Lemma 2, implies a clockwise cycle in $G$; contradicting the leftmost-ness of $L$. If $j < i$, by Part 1 of Lemma 2, $rev(Q) \circ R$ implies a clockwise residual cycle in $G$, contradicting the leftmost-ness of $\mathbf{f}$. $\square$

**Lemma 5.** *There is a maximum $ST$-flow $\mathbf{f}_1$ in $\mathcal{G}_k$ that is obtained from $\mathbf{f}_0$ by removing flow on darts in the first and last $\phi$ copies of $G$ in $\mathcal{G}_k$. Further, the amount of flow into sink $t^i$ for $i \leq k - \phi$ and the amount of flow out of source $s^j$ for $j \geq \phi$ is the same in $\mathbf{f}_0$ and $\mathbf{f}_1$.*

*Proof.* Since $\mathbf{f}_0$ is an acyclic max pseudoflow, it can be converted to a max flow by removing flow from source-to-excess flow paths and deficit-to-sink flow paths [8]. Let $P$ be such a flow path. $P$ must map to a simple path in $G$. By the Pigeonhole Lemma, $P$ must be contained within $\phi$ copies of $G$. This proves the first part of the lemma. Since $P$ cannot start at $s^j$ for $j \geq \phi$ without going through more than $\phi$ copies (and likewise, $P$ cannot end at $t^i$ for $i \leq k - \phi$), the second part of the lemma follows. $\square$

### 2.2 Value of the maximum flow

In the next lemma, we prove that from $\mathbf{f}^{ST}$, the leftmost maximum $ST$-flow in $\mathcal{G}_k^{ST}$, we can extract $|\mathbf{f}|$, the value of the maximum $st$-flow in $G$.

**Lemma 6.** *For $k \geq 4\phi$, the amount of flow through $s^{2\phi}$ in $\mathbf{f}^{ST}$ is $|\mathbf{f}|$.*

*Proof.* We show that the amount of flow leaving $s^{2\phi}$ in $\mathbf{f}^{ST}$ is the same as in $\mathbf{f}_1$. By Lemma 5, the amount of flow leaving $s^{2\phi}$ is the same in $\mathbf{f}_1$ as $\mathbf{f}_0$ which is the same as the amount of flow leaving $s$ in $\mathbf{f}$; this proves the lemma.

First extend $\mathbf{f}_1$ into a (max) $ST$-flow, $\mathbf{f}_1^{ST}$, in $\mathcal{G}_k^{ST}$ in the natural way. To convert $\mathbf{f}_1^{ST}$ into a *leftmost* flow, we must saturate the clockwise residual cycles with a c.w. circulation. By Lemma 1 and for a contradiction, there then must be c.w. simple cycle $C$ that changes the amount of flow through $s^{2\phi}$. $C$ must go through $S$, $C$ is residual w.r.t. $\mathbf{c}_{\mathbf{f}_1^{ST}}$, and cannot visit $T$, therefore $C$ must contain a $s^i$-to-$s^{2\phi}$ residual path $P$ that is in $\mathcal{G}_k$. Since $C$ is c.w. , $i < 2\phi$. Suppose $P$ does not use a dart in the first or last $\phi$ copies of $G$ in $\mathcal{G}_k$. Then $P$ must map to a set $P'$ of darts in $G$ which, by Lemma 5 are residual w.r.t. $\mathbf{c}_\mathbf{f}$. By Part 1 of Lemma 2, $P'$ contains a clockwise cycle, contradicting the leftmostness of $\mathbf{f}$. It follows that $P$ must cross either from the $\phi^{th}$ copy to $s^{2\phi}$ or from $s^{2\phi}$ to the $3\phi^{th}+1$ copy. Then, by the Pigeonhole Lemma, $P$ contains a subpath $Q$ that goes from $\bar{v}$ to $\bar{v}'$, where $\bar{v}$ is an earlier copy of a vertex $v$ than $\bar{v}'$, and neither are in the first or last $\phi$ copies. By Part 1 of Lemma 2, the map of $Q$ contains a clockwise cycle in $G$. Since $Q$ does not contain darts in the first or last $\phi$ copies of $G$, by Lemma 5, this cycle is residual w.r.t. $\mathbf{c}_\mathbf{f}$ in $G$, again contradicting that $\mathbf{f}$ is leftmost. ☐

### 2.3 Maximum flow

Now, suppose we know $|\mathbf{f}|$ (as per Lemma 2.2). We change the capacities of the arcs into $T$ and out of $S$ in $\mathcal{G}_k^{ST}$ to $|\mathbf{f}|$, resulting in capacities $\mathbf{c}^{|\mathbf{f}|}$. Now, $\mathbf{f}_1^{ST}$, respects $\mathbf{c}^{|\mathbf{f}|}$ since, by Lemmas 4 and 5, the amount of flow leaving any source or entering any sink in $\mathbf{f}_1$ is at most $|\mathbf{f}|$. The proof of the following is similar to that of Lemma 6:

**Lemma 7.** *$\mathbf{f}_1^{ST}$ can be converted into a leftmost maximum $ST$-flow $\mathbf{f}^{|\mathbf{f}|}$ for the capacities $\mathbf{c}^{|\mathbf{f}|}$ while not changing the flow on darts in the first or last $2\phi$ copies of $G$ in $\mathcal{G}_k$.*

To summarize, Lemmas 5 and 7 guarantee that the maximum leftmost $ST$-flow, $\mathbf{f}^{|\mathbf{f}|}$, in $\mathcal{G}_k^{ST}$ given capacities $\mathbf{c}^{|\mathbf{f}|}$ has the same flow assignment on the darts in copy $2\phi+1$ as $\mathbf{f}$ so long as $k \geq 4\phi+1$. Starting from scratch, we can find c.w. acyclic capacities $\mathbf{c}$ via Khuller, Naor and Klein's method (one shortest path computation); we can find $|\mathbf{f}|$ (Lemma 6, a second shortest path computation) and then $\mathbf{f}$ (Lemma 7, a third shortest path computation). Therefore, finding a maximum $st$-flow in a directed planar graph $G$ is equivalent to three shortest path computations: one in $G$ and two in a covering of $G$ that is $4\phi+1$ times larger than $G$.

### 2.4 Discussion

The linear bound on the number of augmentations required by Borradaile and Klein's leftmost augmenting-paths algorithm is given by way on an *unusability theorem* which states that an arc can be augmented, and then its reverse can be augmented, but, if this reverse-augmentation occurs, the arc cannot be augmented again. In a companion paper, we show how to implement an augmenting paths algorithm whose analysis depends on a similar unusability theorem using only priority-queue operations [3]. In this algorithm, we also use dual shortest-paths to illustrate the priority-queue implementation. We believe that combining these ideas – the unusability theorem and dual-shortest paths correspondence – could lead to a max *st*-flow algorithm for planar graphs that uses $O(n)$ (instead of $O(\phi n)$ as implied by our work here) priority-queue operations. Provided the constants are reasonable, this would certainly be more efficient in practice than a dynamic-trees based implementation.

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